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On the guaranteed convergence of the fourth order simultaneous method for polynomial zeros

M.S. Petković *, L. Rančić

Faculty of Electronic Engineering, University of Niš, P.O. Box 73, 18 000 Niš, Serbia, Yugoslavia

Abstract

One of the most important and challenging problems in solving nonlinear equations is the construction of computationally verifiable initial conditions that provide both the guaranteed and fast convergence of the considered numerical method. A suitable convergence procedure, based partially on Smale's "point estimation theory" from 1981, is applied in this paper to the fourth order iterative method for the simultaneous approximation of simple zeros of polynomials, proposed by Zheng and Sun in 1999. We have stated initial conditions which ensure the guaranteed convergence of this method. These conditions are of significant practical importance since they depend only on available data: the coefficients of a given polynomial, its degree n and initial approximations to polynomial zeros.

Keywords: Polynomial zeros; Approximate zeros; Simultaneous methods; Point estimation; Initial conditions for convergence

1. Introduction

The construction of computationally verifiable initial conditions that enables both the guaranteed and fast convergence of the considered numerical algorithm is one of the most important and challenging problems in solving

* Corresponding author.
E-mail addresses: msp@junis.ni.ac.yu, msp@eunet.yu (M.S. Petković).

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nonlinear equations. There are a lot of results in the literature concerning convergence analysis which deal with unattainable data. Most frequently, the convergence theorems deal with the sought zeros of an equation, suitable (but unknown) constants or sufficiently close approximations (without a proper estimate of their closeness). These results are rather of theoretical importance.

A practically applicable approach assumes such initial conditions which should depend only on available features of a given function f and initial approximations $z^{(0)}$. Smale's investigations [1] from 1981, known as "point estimation theory" and connected with the safe convergence of Newton's method, was an important advance in this direction. After this result and another Smale's fundamental work [2], the research in this topic has been widely extended by many authors who treated various methods for solving nonlinear equations as well as simultaneous methods for finding polynomial zeros. More details may be found in the book [3], the survey paper [4] and the references cited there.

Let

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 \quad (a_i \in \mathbb{C})$$

be a monic complex algebraic polynomial having only simple zeros $\zeta_1, \ldots, \zeta_n$. Recently Zheng and Sun [5] constructed the fourth order iterative method for the simultaneous determination of complex zeros $\zeta_1, \ldots, \zeta_n$ of P. The authors gave in [5] an outline of the proof of the convergence rate of the proposed method, but without any conditions for the convergence and a domain of convergence. In this paper we state initial conditions which guarantee the convergence of Zheng–Sun's method. These conditions are computationally verifiable, which is of considerable practical importance; namely, they depend only on initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ to the zeros $\zeta_1, \ldots, \zeta_n$ of a polynomial P, its degree and the polynomial coefficients $a_0, a_1, \ldots, a_{n-1}$.

Throughout this paper we will always assume that the polynomial degree n is $\geqslant 3$. For $m = 0, 1, \ldots$, let

$$d^{(m)} = \min_{j \neq i} \left| z_i^{(m)} - z_j^{(m)} \right|$$

be the minimal distance between approximations obtained in the mth iteration, and let

$$W_i^{(m)} = \frac{P(z_i^{(m)})}{\prod_{\substack{j=1 \\ j \neq i}}^{n} \left(z_i^{(m)} - z_j^{(m)}\right)}, \quad w^{(m)} = \max_{1 \leqslant j \leqslant n} |W_j^{(m)}|.$$

Results presented in many papers (see, e.g., [4,6–12]) showed that suitable initial conditions, providing a guaranteed convergence of a wide class of iterative methods for the simultaneous finding polynomial zeros, are of the form of the inequality

$$w^{(0)} < c_n d^{(0)}, \tag{1}$$

where c_n is the quantity which depends only on the polynomial degree n. Let us emphasize that (1) arose in a quite natural way from the convergence analysis of Newton's method presented in [11] by Wang and Zhao.

The minimal distance $d^{(0)} = \min_{j \neq i} |z_i^{(0)} - z_j^{(0)}|$ may be regarded as a measure of separation of initial approximations, while the quantity $w^{(0)} = \min_{i \neq j} |P(z_i^{(0)})/\prod_{j \neq i}(z_i^{(0)} - z_j^{(0)})|$ determines, to a certain sense, the closeness of initial approximations to the wanted zeros. Obviously, the factor c_n should be as great as possible since a greater c_n permits a greater $w^{(0)}$ and, thus, the selection of cruder initial approximations. An extensive discussion about initial conditions of the form (1) may be found in [4,8]. We note that the minimal distance $\min_{j \neq i} |\zeta_i - \zeta_j|$ among zeros instead of (computationally verifiable) distance $d^{(0)}$ was often used in literature, which obviously does not offer a possibility of computational verification of initial conditions.

In Section 3 we present the convergence theorem which provides very simple verification of the convergence of a rather wide class of iterative methods for the simultaneous approximation of polynomial zeros under a given initial condition of the form (1). This theorem is applied to a new method for the simultaneous determination of simple complex zeros of a polynomial, proposed recently by Zheng and Sun in [5] and given briefly in Section 2. For this method initial conditions which enable a guaranteed convergence are stated in Section 3. These conditions are of a practical importance since it depends only on available features of a polynomial and initial approximations.

2. Fourth order simultaneous methods

For some reasonably close approximations $z_1, \ldots, z_n$ to the zeros of P let us introduce

$$W_i = \frac{P(z_i)}{\prod_{\substack{j=1 \\ j \neq i}}^{n} (z_i - z_j)}, \qquad G_{k,i} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{W_j}{(z_i - z_j)^k} \quad (k = 1, 2).$$

Starting from the cubically convergent Hansen–Patrick's method [13], the following one parameter family of iterative methods for the simultaneous approximation of all simple zeros of a polynomial P has been derived in [14]:

$$\hat{z}_i = z_i - \frac{(\alpha + 1) W_i}{\alpha(1 + G_{1,i}) + [(1 + G_{1,i})^2 + 2(\alpha + 1) W_i G_{2,i}]_*^{1/2}} \quad (i \in I_n). \tag{2}$$

Here z_i is a current approximation and $\hat{z}_i$ is a new approximation to the wanted zero. The symbol $*$ points to the choice of the "proper" sign in front of the square root in (2). It has been proved in [14] that the order of convergence of the iterative methods of the family (2) is equal to four for any fixed and finite parameter α. A number of numerical experiments showed that the proposed methods possess good convergence properties.

In particular, for $\alpha = 0$ we obtain from (2) *Ostrowski-like method*, while the choice $\alpha = 1/(n-1)$ yields *Laguerre-like method*. Taking $a = 1$ in (2) we get *Euler-like method*,

$$\hat{z}_i = z_i - \frac{2W_i}{1 + G_{1,i} + [(1 + G_{1,i})^2 + 4W_i G_{2,i}]_*^{1/2}} \quad (i \in I_n). \tag{3}$$

A limiting process when $a \to -1$ in (2) leads to *Halley-like method*,

$$\hat{z}_i = z_i - \frac{W_i}{1 + G_{1,i} + \frac{W_i G_{2,i}}{1+G_{1,i}}} \quad (i \in I_n). \tag{4}$$

The names come from the methods from which the aforementioned methods were derived using a special procedure described in [14]. Let us note that Ellis and Watson [15] constructed the iterative formula (4) using a different approach.

Let us consider the family (2) under the assumption that $|P(z_i)|$ is small enough to provide the inequality

$$|G_{1,i}^2 + 2G_{1,i} + 2(\alpha + 1)W_i G_{2,i}| < 1.$$

Then, using the approximation $\sqrt{1+z} \approx 1 + (z/2)(|z| < 1)$, we obtain from (2)

$$\hat{z}_i = z_i - \frac{W_i}{1 + G_{1,i} + W_i G_{2,i}}. \tag{5}$$

This formula was derived in [5] starting from Euler's formula (3) which is a special case of the family (2). Assuming that $m = 0, 1, 2, \ldots$ is the iteration index, the above formula leads to the following iterative method

$$z_i^{(m+1)} = z_i^{(m)} - \frac{W_i^{(m)}}{1 + G_{1,i}^{(m)} + W_i^{(m)} G_{2,i}^{(m)}} \quad (i \in I_n;\ \ m = 0, 1, \ldots). \tag{6}$$

Comparing the iterative formulas (4) and (5), both of the fourth order, we observe that the iterative method (5) is slightly simpler; it requires n divisions less per iteration (and for all n zeros) compared to (4).

3. Convergence analysis

Before stating the main result concerned with the guaranteed convergence of the simultaneous method (6), we give a general theorem which can be applied to a wide class of simultaneous methods of the form

$$z_i^{(m+1)} = z_i^{(m)} - C_i\left(z_1^{(m)}, \ldots, z_n^{(m)}\right) \quad (i \in I_n;\ \ m = 0, 1, \ldots), \tag{7}$$

where $I_n = \{1, \ldots, n\}$ is the index set and $z_1^{(m)}, \ldots, z_n^{(m)}$ are some distinct approximations to simple zeros $\zeta_i, \ldots, \zeta_n$ respectively, obtained in the mth iterative step by the method (7). In what follows the term

$$C_i^{(m)} = C_i\left(z_1^{(m)}, \ldots, z_n^{(m)}\right) \quad (i \in I_n)$$

will be called the *correction*. For simplicity, we will omit sometimes the iteration index m and denote quantities in the latter $(m+1)$st iteration by $\widehat{}$ ("hat").

Let $\Lambda(\zeta_i)$ be a reasonably close neighborhood of the zero ζ_i $(i \in I_n)$ and let the corrections C_i occurring in (7) can be represented as

$$C_i(z_1, \ldots, z_n) = \frac{P(z_i)}{F_i(z_1, \ldots, z_n)} \quad (i \in I_n), \tag{8}$$

where the function $(z_1, \ldots, z_n) \mapsto F_i(z_1, \ldots, z_n)$ satisfies the following conditions for each $i \in I_n$:

1° $F_i(\zeta_1, \ldots, \zeta_n) \neq 0$,
2° $F_i(z_1, \ldots, z_n) \neq 0$ for distinct approximations $z_i \in \Lambda(\zeta_i)$,
3° $F_i(z_1, \ldots, z_n)$ is continuous in $\mathbb{C}^n$.

Let us define a real function $t \mapsto g(t)$ on the open interval $(0, 1)$ by

$$g(t) = \begin{cases} 1 + 2t, & 0 < t \leqslant \dfrac{1}{2}, \\ \dfrac{1}{1-t} & \dfrac{1}{2} < t < 1. \end{cases}$$

The following theorem (see [4,9]), involving corrections C_i and the function g, plays the key role in our convergence analysis of the simultaneous method (6).

Theorem 1. *Let the iterative method* (7) *have the correction term of the form* (8) *for which the conditions* 1°–3° *hold, and let* $z_1^{(0)}, \ldots, z_n^{(0)}$ *be distinct initial approximations to the zeros of P. If there exists a real number* $\beta \in (0, 1)$ *such that the following two inequalities*

(i) $|C_i^{(m+1)}| \leqslant \beta |C_i^{(m)}| \quad (m = 0, 1, \ldots)$,
(ii) $|z_i^{(0)} - z_j^{(0)}| > g(\beta)\Big(|C_i^{(0)}| + |C_j^{(0)}|\Big) \quad (i \neq j,\ \ i, j \in I_n)$
are valid, then the iterative method (7) *is convergent.*

Let us return to the iterative methods (6) and introduce

$$H_i = 1 + G_{1,i} + W_i G_{2,i}.$$

Then the correction C_i may be written in the form

$$C_i = \frac{W_i}{1 + G_{1,i} + W_i G_{2,i}} = \frac{W_i}{H_i} = \frac{P(z_i)}{F_i(z_1, \ldots, z_n)},$$

where

$$F(z_1, \ldots, z_n) = H_i \prod_{j \neq i} (z_i - z_j) \quad (i \in I_n). \tag{9}$$

The iterative formula (6) becomes (omitting the iteration index)

$$\hat{z}_i = z_i - C_i = z_i - \frac{W_i}{H_i}. \tag{10}$$

Before establishing the main convergence theorem, we give three lemmas which are concerned with some necessary bounds and estimates.

Lemma 1. *Let $z_1, \ldots, z_n$ be distinct approximations to the zeros $\zeta_1, \ldots, \zeta_n$ of a polynomial P, and let $\hat{z}_1, \ldots, \hat{z}_n$ be new approximations obtained by the iterative formula* (10). *If the inequality*

$$w < c_n d, \quad c_n = \frac{1}{2n + 1.3} \quad (n \geqslant 3), \tag{11}$$

holds, then for all $i \in I_n$ we have

(i) $\lambda_n := c_n/(1 - (n-1)c_n(1 + c_n)) < (1/5)$;
(ii) $c_n/\lambda_n < |H_i| < 1 + (n-1)c_n(1 + c_n) =: M_n$;
(iii) $|\hat{z}_i - z_i| < (\lambda_n/c_n)|W_i| < \lambda_n d$.

Proof. The sequence $\{\lambda_n\}$ is monotonically decreased so that $\lambda_n \leqslant \lambda_3 = 0.1989 \cdots < 1/5$ and (i) is proved.

Since

$$|G_{1,i}| = \left| \sum_{j \neq i} \frac{W_j}{z_i - z_j} \right| \leqslant \sum_{j \neq i} \frac{|W_j|}{|z_i - z_j|} \leqslant \frac{(n-1)w}{d} < (n-1)c_n,$$

$$|G_{2,i}| = \left| \sum_{j \neq i} \frac{W_j}{(z_i - z_j)^2} \right| \leqslant \sum_{j \neq i} \frac{|W_j|}{|z_i - z_j|^2} \leqslant \frac{(n-1)w}{d^2} < \frac{(n-1)c_n}{d},$$

we have

$$\begin{aligned} |H_i| &\geqslant 1 - |G_{1,i}| - |W_i||G_{2,i}| > 1 - (n-1)c_n - \frac{w(n-1)c_n}{d} \\ &> 1 - (n-1)c_n - (n-1)c_n^2 = 1 - (n-1)c_n(1 + c_n) = \frac{c_n}{\lambda_n}, \end{aligned}$$

and

$$\begin{aligned} |H_i| &\leqslant 1 + |G_{1,i}| + |W_i||G_{2,i}| \\ &< 1 + (n-1)c_n + (n-1)c_n^2 = 1 + (n-1)c_n(1 + c_n) = M_n, \end{aligned}$$

which proves (ii).

From the iterative formulas (10) we have

$$\hat{z}_i - z_i = -C_i = -\frac{W_i}{H_i},$$

wherefrom

$$|\hat{z}_i - z_i| \leqslant \frac{|W_i|}{|H_i|} < \frac{\lambda_n}{c_n}|W_i| < \lambda_n d, \tag{12}$$

which completes the proof of (iii). □

Lemma 2. *For distinct complex numbers* $z_1, \ldots, z_n$, $\hat{z}_1, \ldots, \hat{z}_n$ *let*

$$d = \min_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} |z_i - z_j|, \qquad \hat{d} = \min_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} |\hat{z}_i - \hat{z}_j| \quad (i \in I_n).$$

If (11) *holds, then*

$$|\hat{z}_i - z_j| > (1 - \lambda_n)d \quad (i, j \in I_n), \tag{13}$$

$$|\hat{z}_i - \hat{z}_j| > (1 - 2\lambda_n)d \quad (i, j \in I_n), \tag{14}$$

and

$$\left| \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{\hat{z}_i - z_j}{\hat{z}_i - \hat{z}_j} \right| < \left(1 + \frac{\lambda_n}{1 - 2\lambda_n}\right)^{n-1}. \tag{15}$$

The assertions of this lemma are easy to prove by using the triangle inequality and the inequality (12). Let us note that the necessary condition $\lambda_n < 1/2$ is fulfilled under the condition (11).

Remark 1. Since (14) is valid for arbitrary pair $i, j \in I_n$ and $\lambda_n < 1/5 < 1/2$ if (11) holds, it follows

$$\hat{d} = \min_{j \neq i} |\hat{z}_i - \hat{z}_j| > (1 - 2\lambda_n)d. \tag{16}$$

Lemma 3. *Let distinct approximations* $z_1, \dots, z_n$ *satisfy the conditions* (11) *and let*

$$\delta_n := \left((n-1)c_n\lambda_n + \frac{(n-1)\lambda_n^2}{1-\lambda_n}\right)\left(1 + \frac{\lambda_n}{1-2\lambda_n}\right)^{n-1},$$
$$\bar{\delta}_n := \frac{(1-2\lambda_n)c_n}{M_n\lambda_n},$$
$$\beta_n := \frac{M_n\lambda_n\delta_n}{c_n}.$$

Then

(i) $\delta_n < \bar{\delta}_n < 1 - 2\lambda_n < \frac{3}{5}$;
(ii) $\beta_n < 1 - 2\lambda_n < \frac{3}{5}$;
(iii) $|\widehat{W}_i| < \delta_n |W_i|$;
(iv) $\widehat{w} < c_n \hat{d}$.

Proof. Substituting λ_n and M_n in the expression for $\bar{\delta}_n$, we find

$$\bar{\delta}_n = (1 - 2\lambda_n) \cdot \frac{1 - (n-1)c_n(1 + c_n)}{1 + (n-1)c_n(1 + c_n)} < 1 - 2\lambda_n < \frac{3}{5}.$$

Using a tedious but elementary analysis we infer that the sequence $\{\bar{\delta}_n - \delta_n\}_{n=3,4,\dots}$ is positive for any $n \geqslant 3$, which furnishes the left inequality

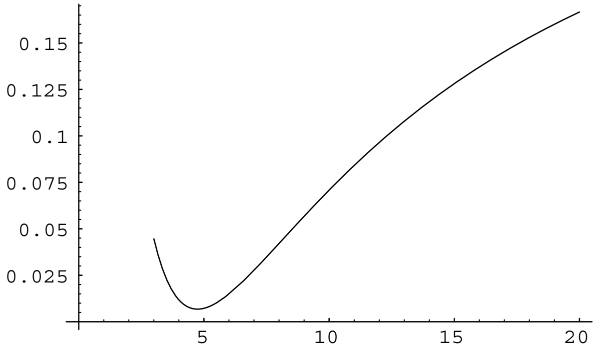

Fig. 1. The sequence $\{\bar{\delta}_n - \delta_n\}$.

in (i). The behavior of this sequence is shown graphically in Fig. 1 by a continuous curve.

By (i) we obtain

$$\beta_n = \frac{M_n \lambda_n \delta_n}{c_n} < \frac{M_n \lambda_n \bar{\delta}_n}{c_n} = 1 - 2\lambda_n < \frac{3}{5}$$

and (ii) is proved. The sequence $\{1 - 2\lambda_n - \beta_n\}_{n=3,4,\dots}$ is displayed in Fig. 2.

For distinct points $z_1, \dots, z_n$ let us define the polynomial Q of degree n by

$$Q(z) = \prod_{j=1}^{n} (z - z_j).$$

Applying Heaviside's development into partial fractions we find

$$\frac{P(z) - Q(z)}{Q(z)} = \sum_{j=1}^{n} \frac{A_j}{z - z_j}, \quad \text{where } A_j = \frac{P(z_j) - Q(z_j)}{Q'(z_j)} = \frac{P(z_j)}{Q'(z_j)} = W_j.$$

Hence we obtain the following representation of the polynomial P:

$$P(z) = \left(\sum_{j=1}^{n} \frac{W_j}{z - z_j} + 1 \right) Q(z) = \left(\sum_{j=1}^{n} \frac{W_j}{z - z_j} + 1 \right) \prod_{j=1}^{n} (z - z_j).$$

Putting $z = \hat{z}_i$ in the last relation we find

$$P(\hat{z}_i) = (\hat{z}_i - z_i) \left(\frac{W_i}{\hat{z}_i - z_i} + \sum_{j \neq i} \frac{W_j}{\hat{z}_i - z_j} + 1 \right) \prod_{j \neq i} (\hat{z}_i - z_j).$$

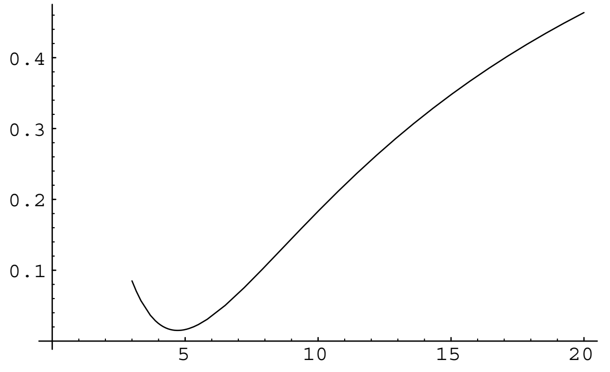

Fig. 2. The sequence $\{1 - 2\lambda_n - \beta_n\}$.

After dividing with $\prod_{j\neq i}\left(\hat{z}_i-\hat{z}_j\right)$, we obtain

$$\widehat{W}_i=\frac{P(\hat{z}_i)}{\prod_{j\neq i}\left(\hat{z}_i-\hat{z}_j\right)}$$
$$=(\hat{z}_i-z_i)\left(\frac{W_i}{\hat{z}_i-z_i}+\sum_{j\neq i}\frac{W_j}{\hat{z}_i-z_j}+1\right)\prod_{j\neq i}\left(1+\frac{\hat{z}_j-z_j}{\hat{z}_i-\hat{z}_j}\right). \tag{17}$$

From the iterative formula (10) we have

$$\frac{W_i}{\hat{z}_i-z_i}=-H_i=-1-G_{1,i}-W_iG_{2,i}. \tag{18}$$

Substituting (18) in (17) one obtains

$$\widehat{W}_i=-(\hat{z}_i-z_i)\left(W_iG_{2,i}+(\hat{z}_i-z_i)\sum_{j\neq i}\frac{W_j}{(\hat{z}_i-z_j)(z_i-z_j)}\right)\prod_{j\neq i}\left(1+\frac{\hat{z}_j-z_j}{\hat{z}_i-\hat{z}_j}\right),$$

wherefrom

$$|\widehat{W}_i|\leqslant|\hat{z}_i-z_i|\left(|W_iG_{2,i}|+|\hat{z}_i-z_i|\sum_{j\neq i}\frac{|W_j|}{|\hat{z}_i-z_j||z_i-z_j|}\right)\prod_{j\neq i}\left(1+\frac{|\hat{z}_j-z_j|}{|\hat{z}_i-\hat{z}_j|}\right). \tag{19}$$

Since

$$|W_iG_{2,i}|<|W_i|\frac{(n-1)w}{d^2}<(n-1)c_n^2$$

and

$$|\hat{z}_i-z_i|\sum_{j\neq i}\frac{|W_j|}{|\hat{z}_i-z_j||z_i-z_j|}<\lambda_n d\,\frac{(n-1)w}{(1-\lambda_n)d^2}<\frac{(n-1)\lambda_n c_n}{1-\lambda_n}$$

(by (11), (13) and Lemma 1(iii)), from (iii) of Lemma 1 and (19) one gets

$$|\widehat{W}_i|<\frac{\lambda_n}{c_n}|W_i|\left((n-1)c_n^2+\frac{(n-1)\lambda_n c_n}{1-\lambda_n}\right)\left(1+\frac{\lambda_n}{1-2\lambda_n}\right)^{n-1}$$
$$=|W_i|\left((n-1)c_n\lambda_n+\frac{(n-1)\lambda_n^2}{1-\lambda_n}\right)\left(1+\frac{\lambda_n}{1-2\lambda_n}\right)^{n-1}=\delta_n|W_i|.$$

Therefore, the assertion (iii) is valid.

To prove (iv) we use (11), (16) and the assertions (i) and (iii) of Lemma 3 to obtain

$$|\widehat{W}_i|<\delta_n|W_i|<(1-2\lambda_n)c_nd<c_n\hat{d}. \qquad \square$$

Now we are able to establish the main convergence theorem for the iterative method (6).

Theorem 2. *If the initial approximations* $z_1^{(0)}, \dots, z_n^{(0)}$ *satisfy the initial condition*

$$w^{(0)} < c_n d^{(0)}, \tag{20}$$

then the iterative method (6) *is convergent.*

Proof. It is sufficient to prove that the inequalities (i) and (ii) of Theorem 1 are valid for the correction $C_i^{(m)} = W_i^{(m)}/H_i^{(m)}$ which appears in the considered method (6).

By virtue of Lemma 3, whose assertions are valid because (20) holds, we can prove by induction that

$$w^{(m+1)} < \delta_n w^{(m)} < (1-2\lambda_n)c_n d^{(m)} < c_n d^{(m+1)}$$

holds for each $m = 0, 1, \dots$

Starting from assertion (ii) of Lemma 1, under the condition (20), for the function F_i appearing in (8) and (9) we prove by induction

$$\left|F_i\left(z_1^{(m)}, \dots, z_n^{(m)}\right)\right| = \left|H_i^{(m)}\right| \prod_{j \neq i} \left|z_i^{(m)} - z_j^{(m)}\right| > \frac{c_n}{\lambda_n}[d^{(m)}]^{n-1} > 0$$

for each $i \in I_n$ and $m = 0, 1, \dots$ Therefore, the iterative method (6) is well defined in each iteration.

Using Lemmas 1(ii) and 3(iii), we find

$$|\widehat{C}_i| = \frac{|\widehat{W}_i|}{|H_i|} < \frac{\lambda_n}{c_n}|\widehat{W}_i| < \frac{\lambda_n \delta_n}{c_n}|W_i| = \frac{\lambda_n \delta_n}{c_n}\frac{|W_i|}{|H_i|}|H_i| < \frac{\lambda_n \delta_n}{c_n}|C_i|M_n = \beta_n|C_i|,$$

where $\beta_n < 3/5$ (Lemma 3(ii)). Using the same argumentation we prove by induction

$$|C_i^{(m+1)}| < \beta_n |C_i^{(m)}|$$

for each $i \in I_n$ and $m = 0, 1, \dots$ Therefore, the inequality (i) of Theorem 1 is valid.

According to Lemma 3(ii) we see that $\delta_n \in (0, 3/5) \subset (0, 1)$ so that the function g appearing in Theorem 1 is defined. Having in mind the definition of the function g, for $\beta_n \leqslant 1/2$ we find by Lemma 3(ii)

$$g(\beta_n) = 1 + 2\beta_n < 1 + 2(1-2\lambda_n) = 3 - 4\lambda_n < \frac{1}{2\lambda_n}$$

since $\lambda_n < 1/5$ (Lemma 1(i)). If $1/2 < \beta_n < 3/5$, then it follows

$$g(\beta_n) = \frac{1}{1-\beta_n} < \frac{1}{1-(1-2\lambda_n)} = \frac{1}{2\lambda_n}$$

(Lemma 3(ii)). Therefore, the inequality

$$g(\beta_n) < \frac{1}{2\lambda_n} \tag{21}$$

holds under the condition (20).

By Lemma 1(ii) and (20) we estimate

$$\frac{1}{\lambda_n}|C_i^{(0)}| < \frac{|W_i^{(0)}|}{c_n} < d^{(0)} \text{ for each } i \in I_n. \tag{22}$$

According to (21) and (22) we obtain

$$\left|z_i^{(0)} - z_j^{(0)}\right| \geqslant d^{(0)} > \frac{w^{(0)}}{c_n} > \frac{1}{2\lambda_n}\left(\left|C_i^{(0)}\right| + \left|C_j^{(0)}\right|\right) > g(\beta_n)\left(\left|C_i^{(0)}\right| + \left|C_j^{(0)}\right|\right)$$

for each $i \neq j$, $i, j \in I_n$. This proves (ii) of Theorem 1. The validity of (i) and (ii) of Theorem 1 shows that the iterative method (6) is convergent under the given condition (20). □

Remark 2. The constant c_n in (20) has been chosen to be almost optimal. To realize this aim, we employed an improving procedure which consists of refinement of the inequalities

$$\delta_n < \bar{\delta}_n \ (< 1 - 2\lambda_n), \quad \beta_n < 1 - 2\lambda_n, \quad g(\beta_n) < \frac{1}{2\lambda_n}$$

to be as sharp as possible. For this purpose, the program *Mathematica* 4.1 was used.

Remark 3. As mentioned in Remark 2, the inequalities

$$\bar{\delta}_n > \delta_n, \quad 1 - 2\lambda_n > \beta_n,$$

have to be satisfied for every $n \geqslant 3$. The range [3,8] is a critical region for the validity of these inequalities, as shown in Figs. 1 and 2. This fact forces the choice of the factor c_n to be not greater than $1/(2n + 1.3)$ (assuming that the constants $a = 2$ and $b = 1.3$ in $c_n = 1/(an + b)$ are rounded to the first decimal digit).

However, outside this critical range (for $n \geqslant 9$) we can choose a greater c_n, which is desirable. We again apply a refinement procedure and come to the constant $c_n = 1/(2n + 0.8)$ $(n \geqslant 9)$. Therefore, slightly improved initial conditions read thus:

$$c_n = \begin{cases} \dfrac{1}{2n+1.3}, & \text{if } 3 \leqslant n \leqslant 8, \\ \dfrac{1}{2n+0.8}, & \text{if } n \geqslant 9. \end{cases}$$

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