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A QUADRATICALLY CONVERGENT ITERATIVE METHOD FOR NONLINEAR EQUATIONS

Beong In Yun and Miodrag S. Petković

Abstract. In this paper we propose a simple iterative method for finding a root of a nonlinear equation. It is shown that the new method, which does not require any derivatives, has a quadratic convergence order. In addition, one can find that a hybrid method combined with the non-iterative method can further improve the convergence rate. To show the efficiency of the presented method we give some numerical examples.

1. Introduction

Until recently lots of iterative methods for solving a nonlinear equation $f(x) = 0$ have been proposed [1, 3, 4, 5, 6, 7, 9, 12, 14]. Most of these methods are based on the Newton method or the secant method, and an extensive analysis of numerical results and some valuable remarks are included in [9]. It should be noted that the availability of most iterative methods based on the Newton method depends on an initial guess and behavior of the function $f(x)$ near the root. Moreover, an explicit form of the derivative $f'(x)$ is necessary in implementing the Newton method. Though the secant method can overcome this problem, it takes a cost of slower rate of convergence.

To get over the difficulties such as the choice of initial guess and improper behavior of $f(x)$ in using the existing iterative methods, one of the authors recently proposed a non-iterative method in the work [13]. The method is based on a transform of $f(x)$ via a hyperbolic tangent function or a signum function. Then numerical evaluation of the integration of the transformed function should be performed to find an approximate root, directly. The error of this method depends only on the accuracy of the numerical integration. In order to obtain sufficiently accurate numerical integration, however, a large number of

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integration points are needed because the integrand or the transformed function is a step-like function.

The purpose of this work is to develop a new simple iterative method to remedy the aforementioned drawbacks of the Newton method and the secant method. As a result, the proposed method maintains quadratic convergence without requiring a derivative of the function $f(x)$ nor the effort to choose a proper initial guess. Furthermore, to accelerate convergence rate, one may replace an initial interval using the non-iterative method proposed in [13].

2. A new iterative method

Under the assumption that a continuous function $f(x)$ has a unique zero on an interval $[a,b]$ with $f(a)f(b) < 0$, let $x_k \in (a,b)$ be an approximation to a root p of an equation $f(x) = 0$. Set $a_k = x_k - h_k$ and $b_k = x_k + h_k$ so that p is included in the subinterval $[a_k, b_k]$ (or $[b_k, a_k]$) of $[a,b]$ for some $h_k \neq 0$.

Denote by $L(p;x)$ and $L(x_k;x)$ two piecewise linear functions as

$$L(p;x) := \begin{cases} \dfrac{f(a_k)}{a_k - p}\,(x-p)\,, & x \le p \\ \dfrac{f(b_k)}{b_k - p}\,(x-p)\,, & x > p \end{cases} \tag{2.1}$$

and

$$L(x_k;x) := \begin{cases} \dfrac{f(a_k) - f(x_k)}{a_k - x_k}\,(x-x_k) + f(x_k)\,, & x \le x_k \\ \dfrac{f(b_k) - f(x_k)}{b_k - x_k}\,(x-x_k) + f(x_k)\,, & x > x_k\,. \end{cases} \tag{2.2}$$

Actually, $L(p;x)$ and $L(x_k;x)$ interpolate $f(x)$ at the points $x = a_k,\ p,\ b_k$ and $x = a_k,\ x_k,\ b_k$, respectively (see Figure 1). We define two integrals,

$$\begin{aligned} I_k(p) &:= \int_{a_k}^{b_k} L(p;x)\,dx = \frac{1}{2}\left\{(p-a_k)f(a_k) + (b_k - p)f(b_k)\right\} \\ &= \frac{1}{2}\left\{(p-x_k)\left[f(a_k) - f(b_k)\right] + h_k\left[f(a_k) + f(b_k)\right]\right\} \end{aligned} \tag{2.3}$$

and

$$J_k := \int_{a_k}^{b_k} L(x_k;x)\,dx = \frac{h_k}{2}\left\{2f(x_k) + \left[f(a_k) + f(b_k)\right]\right\}\,. \tag{2.4}$$

The integrals $I_k(p)$ and J_k become closer as the radius $|h_k|$ of the interval $[a_k, b_k]$ (or $[b_k, a_k]$) is decreasing. Thus, replacing p by x_{k+1} in (2.3) and solving the equation $I_k(x_{k+1}) = J_k$, we have an iteration formula as follows: For an initial guess $x_0 = (a+b)/2$,

$$x_{k+1} = x_k - \left[\frac{2h_k}{f(b_k) - f(a_k)}\right] f(x_k)\,, \quad k = 0, 1, 2, \ldots, \tag{2.5}$$

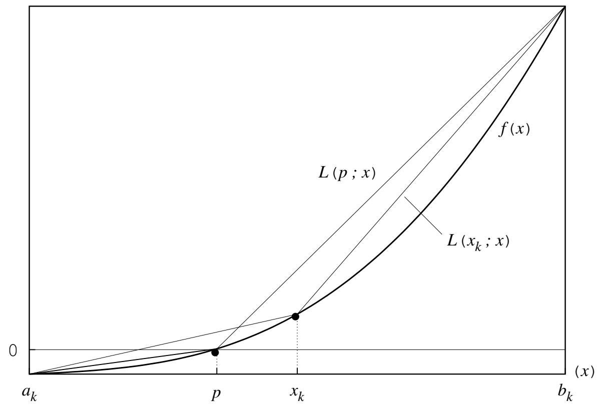

FIGURE 1. Graphs of the lines $L(p;x)$ and $L(x_k;x)$ over an interval (a_k,b_k) centered on an approximation x_k to a zero p of $f(x)$, where $a_k = x_k - h_k$ and $b_k = x_k + h_k$.

where

$$a_k = x_k - h_k\,, \quad b_k = x_k + h_k$$

and $h_k := x_k - x_{k-1}$, $k \geq 1$, with $h_0 = (b-a)/2$ (Note that $a_0 = a$, $b_0 = b$).

The method (2.5) seems to be a variant of the secant method which requires an additional function evaluation at $b_k = 2x_k - x_{k-1}$, in each iteration, compared with the traditional secant method

$$x_{k+1} \;=\; x_k \;-\; \left[\frac{h_k}{f(x_k)-f(a_k)}\right] f(x_k)\,.$$

Referring to the literature [11], one can see that the presented method with the step-length $h_k = x_k - x_{k-1}$ will have a lower bound $(1+\sqrt{5})/2$ for convergence order. However, the method includes more accurate derivative approximation than the secant method such as

$$\left[\frac{2h_k}{f(b_k)-f(a_k)}\right]^{-1} \;=\; \frac{1}{2}\left\{\frac{f(b_k)-f(x_k)}{h_k} \;+\; \frac{f(x_k)-f(a_k)}{h_k}\right\} \;\approx\; f'(x_k)$$

which is an average of the slopes of two straight lines respectively passing through the points $(a_k, f(a_k))$, $(x_k, f(x_k))$ and $(b_k, f(b_k))$, $(x_k, f(x_k))$ while the secant method includes derivative approximation by a slope of a single line passing through the points $(a_k, f(a_k))$, $(x_k, f(x_k))$. Figure 2 illustrates the proposed method (2.5), geometrically, which shows that the iterate x_{k+1} is a zero of the line L_1 passing through the point $(x_k, f(x_k))$ with the slope $(f(b_k) - f(a_k))\,/2h_k$, that is, parallel with the line L passing through the points $(a_k, f(a_k))$ and $(b_k, f(b_k))$.

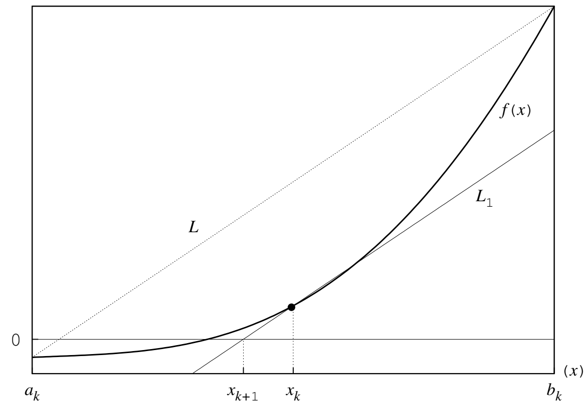

FIGURE 2. Illustration of the presented iterative method, in (2.5), for evaluating x_{k+1} from x_k, $a_k = x_k - h_k$ and $b_k = x_k + h_k$.

In the next section it is proved that, though the presented method has the disadvantage of the additional expense in function evaluations, it has the same convergence order with Newton's method.

3. Convergence analysis

The following theorem implies that the convergence order of the proposed iterative method becomes *quadratic* as the number of the iterations is increasing.

Theorem 3.1. *Let $f(x)$ be twice continuously differentiable on an interval $[a,b]$ and let $f'''(x)$ exist on (a,b). Suppose that $f(x)$ has a unique zero p with $f'(x) \neq 0$ on an interval $[a,b]$. Then the iterative method* (2.5) *is quadratically convergent once an iterate x_k for some k is sufficiently close to p.*

Proof. From the equation (2.5), using the first order Newton's divided difference notation and the equation $f(p) = 0$, we have

$$\begin{aligned} p - x_{k+1} &= p - x_k + \left[\frac{b_k - a_k}{f(b_k) - f(a_k)}\right] f(x_k) \\ &= p - x_k + \frac{f(x_k)}{f[a_k, b_k]} \\ &= \frac{p - x_k}{f[a_k, b_k]} \left\{ f[a_k, b_k] - \frac{f(p) - f(x_k)}{p - x_k} \right\} \\ &= -\frac{(p - x_k)(p - b_k)}{f[a_k, b_k]} \left\{ f[x_k, p] - f[a_k, b_k] \right\} / (p - b_k) \end{aligned}$$

$$= -\frac{(p-x_k)(p-b_k)}{f[a_k,b_k]}\left\{\frac{f[x_k,p]-f[b_k,x_k]}{p-b_k}+\frac{f[b_k,x_k]-f[a_k,b_k]}{p-b_k}\right\}.$$

Introducing the second order Newton's divided difference notation, we have

(3.1)

$$\begin{aligned} p-x_{k+1} &= -\frac{(p-x_k)(p-b_k)}{f[a_k,b_k]}\left\{f[b_k,x_k,p]-\frac{f[b_k,a_k]-f[x_k,b_k]}{a_k-x_k}\cdot\frac{a_k-x_k}{p-b_k}\right\} \\ &= -\frac{(p-x_k)(p-b_k)}{f[a_k,b_k]}\left\{f[b_k,x_k,p]-f[x_k,b_k,a_k]\,\frac{a_k-x_k}{p-b_k}\right\} \\ &= -\frac{(p-x_k)(p-b_k)}{2f'(\xi_k)}\left\{f''(\eta_{k,1})-f''(\eta_{k,2})\,\frac{a_k-x_k}{p-b_k}\right\} \end{aligned}$$

for some ξ_k, $\eta_{k,1}$ and $\eta_{k,2}$ on a neighborhood of p containing the interval $[a_k,b_k]$. In the last equality the following relations were applied.

$$f[\alpha,\beta] = f'(\xi)\,, \quad f[\alpha,\beta,\gamma] = \frac{1}{2}\,f''(\eta)$$

for some ξ between α and β, and η between the minimum and maximum of α, β, γ.

On the other hand, by the mean value theorem for f'',

$$f''(\eta_{k,2}) = f''(\eta_{k,1}) + (\eta_{k,2}-\eta_{k,1})f'''(w_k)$$

for some w_k between $\eta_{k,1}$ and $\eta_{k,2}$. Thus we have

$$\begin{aligned} & f''(\eta_{k,1})-f''(\eta_{k,2})\,\frac{a_k-x_k}{p-b_k} \\ &= f''(\eta_{k,1})-[f''(\eta_{k,1}) + (\eta_{k,2}-\eta_{k,1})f'''(w_k)]\,\frac{a_k-x_k}{p-b_k} \\ &= f''(\eta_{k,1})\,\frac{p-x_k}{p-b_k} - (\eta_{k,2}-\eta_{k,1})f'''(w_k)\frac{a_k-x_k}{p-b_k} \end{aligned}$$

so that (3.1) becomes

(3.2)

$$p-x_{k+1} = -\frac{(p-x_k)^2}{2f'(\xi_k)}\,f''(\eta_{k,1}) - \frac{(p-x_k)}{2f'(\xi_k)}\,\{(a_k-x_k)(\eta_{k,2}-\eta_{k,1})\}\,f'''(w_k)\,.$$

Set $e_k = |p-x_k|$. Then we can see that

$$(3.3)\quad |a_k-x_k| = |h_k| = |x_k-x_{k-1}| = |x_k-p+p-x_{k-1}| \le e_k+e_{k-1}\,.$$

In addition, assuming that $h_k > 0$ without loss of generality, for sufficiently close x_k to the root p there is a neighborhood U_k of p,

$$U_k := [a_k-m_kh_k,\, b_k+m_kh_k]\,,$$

where $m_k = O(1)$ is a positive multiplier, such that

$$p\in U_k\,,\quad \eta_{k,1},\ \eta_{k,2}\in U_k\,,\quad [a_k,b_k]\subset U_k\,.$$

Then

$$
\begin{aligned}
|\eta_{k,2} - \eta_{k,1}| &\le \ (b_k + m_k h_k) \ - \ (a_k - m_k h_k) \\
&= \ (b_k - a_k) \ + \ 2m_k h_k \ = \ 2h_k \ + \ 2m_k h_k \\
&\le \ (2m_k + 2)(e_k + e_{k-1})\,.
\end{aligned} \tag{3.4}
$$

Substituting the inequalities (3.3) and (3.4) into (3.2), we have a relation

$$e_{k+1} \ \le \ c_1 e_k^2 \ + \ c_2 e_k \left(e_k^2 + 2 e_k e_{k-1} + e_{k-1}^2 \right) \tag{3.5}$$

for some positive constants c_1 and c_2. If the errors e_k and e_{k-1} are sufficiently small, it follows that

$$e_{k+1} \ \le \ d_1 e_k^2 \ + \ d_2 e_k e_{k-1}^2 \tag{3.6}$$

for some positive constants d_1 and d_2. Let us assume that the convergence order of the iteration (2.5) is $r > 0$, that is, for some positive constants q_1 and q_2

$$e_{k+1} \ = \ q_1 e_k^r\,, \quad e_k \ = \ q_2 e_{k-1}^r\,.$$

Then (3.6) becomes

$$(q_1 q_2^r)\, e_{k-1}^{r^2} \ \le \ \left(d_1 q_2^2 \right) e_{k-1}^{2r} \ + \ (d_2 q_2)\, e_{k-1}^{r+2}\,,$$

which implies that

$$r^2 \ = \ \min\{2r, r+2\}\,.$$

Therefore we have $r = 2$, and thus the proof of the quadratic convergence is completed. □

4. Numerical examples

Using a programming package, *Mathematica V.6* with 500 digits of precision, we take the following examples $f_i(x) = 0$, $i = 1, 2, \ldots, 7$, to show the availability and efficiency of the proposed iterative method. In particular, we compare the new iterative method with two traditional methods such as the secant method and the Newton method, referring to the fact that the new method takes a similar form to the secant method and it has the same convergence order as the Newton method.

Example 1. $f_1(x) \ = \ x^4 + x - 1\,, \quad 0 \le x \le 2.$

Example 2. $f_2(x) \ = 1 + (x-2)e^{-x}\,, \quad -2 \le x \le 2.$

Example 3. $f_3(x) \ = \ 1 \ - \ (\sin(\pi x/5) - x)^2\,, \quad 0 \le x \le 5.$

Example 4. $f_4(x) \ = \ -\dfrac{x^3 + x - 11}{3x^4 - 2x^2 + 5}\,, \quad 1 \le x \le \dfrac{5}{2}.$

Example 5 (Chen [3]**).** $f_5(x) \ = \ e^{\sin x} - x - 1\,, \quad 1 \le x \le 4.$

Example 6 (Petković [9]**).** $f_6(x) \ = \ x^{40} + x^{39} - 2\,, \quad \dfrac{1}{2} \le x \le 2.$

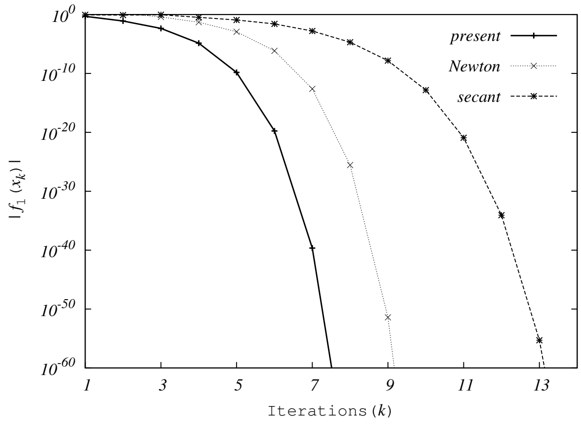

FIGURE 3. Comparison of the numerical errors of the presented method, the secant method, and the Newton method for an equation $f_1(x) = 0$.

Example 7. $f_7(x) = \tan^{-1}(50x) - \dfrac{1}{2}, \quad -2 \le x \le 5.$

Table 1. Numerical results of the presented method (2.5) for the examples $f_i(x) = 0$, $i = 1, 2, 3$.

Iterations	$f_i(x_k)$		
k	$i = 1$	$i = 2$	$i = 3$
1	5.1×10^{-1}	-6.3×10^{-1}	-5.9×10^{-1}
2	8.2×10^{-2}	-9.8×10^{-2}	-8.7×10^{-2}
3	4.5×10^{-3}	-5.3×10^{-3}	-3.8×10^{-3}
4	1.4×10^{-5}	-1.7×10^{-5}	-7.7×10^{-6}
5	1.5×10^{-10}	-1.7×10^{-10}	-3.3×10^{-11}
6	1.7×10^{-20}	-1.7×10^{-20}	-5.9×10^{-22}
7	2.2×10^{-40}	-1.8×10^{-40}	-2.0×10^{-43}
8	3.5×10^{-80}	-1.9×10^{-80}	-2.1×10^{-86}
9	8.9×10^{-160}	-2.2×10^{-160}	-2.4×10^{-172}
10	5.8×10^{-319}	-2.9×10^{-320}	-3.2×10^{-344}

Table 1 includes numerical results of the first ten iterations for Example 1 – Example 3, which implies the quadratic convergence for $k \ge 5$. In addition, for Example 1, Figure 3 illustrates that the convergence rate of the proposed iteration is superior to both the secant method and the Newton method. In the Newton method, throughout this section, the right end-point of the given interval is used as an initial guess.

For all the examples Example 1 – Example 7, numerical results of the presented method are given in Table 2 compared with the existing methods. It includes the values of $|f_i(x_k)|$, with the related iteration number k, in the range of $c_1 \times 10^{-40} \le |f_i(x_k)| \le c_2 \times 10^{-21}$, $1 \le c_1, c_2 < 10$. The term "div."means that the iteration does not satisfy a stopping criterion $|f_i(x_k)| < 10^{-6}$ within the maximum number of iterations $K_{\max} = 10^5$. The number of function evaluations (NFE) is also given, where the cost of $f'(x_k)$ in Newton method is counted as 2. One can see that the new method is superior to the other methods for most examples. Particularly, it should be noted that $f_5(x)$, $f_6(x)$ and $f_7(x)$ have so pathological behavior that most existing iterative methods require very many iterations to produce sufficiently accurate approximation, or they even fail to converge. In contrast, as shown in Table 2, the presented method seems to be still available for these examples.

For the improper cases such as $f_6(x) = 0$ and $f_7(x) = 0$, noting that the convergence of the presented method is rather slow, we can obtain more efficient initial approximation by using the non-iterative method [13, 10]. That is, in the proposed iteration (2.5), we replace the initial approximation x_0 by $\xi_{N,0}$ evaluated as below.

$$\xi_{N,0} = \frac{1}{2}\left\{a + b + \operatorname{sgn}(f(a)) I_N(\operatorname{sgn}(f))\right\}, \tag{4.1}$$

where $I_N(\operatorname{sgn}(f))$ is a numerical integration of $\operatorname{sgn}(f)$ on the given interval (a, b). If we employ the N point trapezoidal rule, the formula (4.1) becomes

$$\xi_{N,0} = \frac{1}{2}\left\{a + b + \operatorname{sgn}(f(a))\frac{b-a}{N}\sum_{j=1}^{N-1}\operatorname{sgn}(f(t_j))\right\} \tag{4.2}$$

for the integration points $t_j = a + j(b-a)/N$. For an exact zero p of $f(x)$, it can be easily seen that $|p - \xi_{N,0}| < \delta = (b-a)/(2N)$, and thus we take an interval

$$(a', b') = (\xi_{N,0} - \delta,\ \xi_{N,0} + \delta), \tag{4.3}$$

which includes the root p. Therefore we may consider a hybrid method in which the interval (a', b'), instead of (a, b), is used as an initial interval in implementing the presented iterative method (2.5). It should be noted that this method requires only $N - 1$ additional evaluations of the sign of $f(x)$ as a preprocess.

Table 2. Numerical results of the presented method, the secant method and the Newton method for the examples $f_i(x) = 0$, $i = 1, 2, \ldots, 7$. The term $(-h)_k$ means $|f_i(x_k)| = O\left(10^{-h}\right)$ in the kth iteration.

Examples	presented method		secant method		Newton method	
	$\lvert f_i(x_k)\rvert$	NFE	$\lvert f_i(x_k)\rvert$	NFE	$\lvert f_i(x_k)\rvert$	NFE

$f_1(x)=0$	$(-40)_7$	14	$(-21)_{11}$	11	$(-26)_8$	24
$f_2(x)=0$	$(-40)_7$	14	$(-22)_{26}$	26	$(-38)_{14}$	42
$f_3(x)=0$	$(-22)_6$	12	$(-29)_{21}$	21	$(-28)_8$	24
$f_4(x)=0$	$(-35)_7$	14	$(-28)_{11}$	11	$(-21)_7$	21
$f_5(x)=0$	$(-25)_6$	12	$(-23)_{11}$	11	$(-21)_{33}$	99
$f_6(x)=0$	$(-21)_{16}$	32	div.	–	$(-39)_{33}$	99
$f_7(x)=0$	$(-34)_{27}$	54	div.	–	div.	–

In Table 3, for the examples $f_6(x)=0$ and $f_7(x)=0$, numerical results of the hybrid methods associated with the presented method, the Newton method and the Ostrowski method [8] of order 4 are included. The Ostrowski method is given by

$$x_{k+1}=x_k-u_k+u_k\,\frac{f(x_k-u_k)}{2f(x_k-u_k)-f(x_k)},\quad u_k=\frac{f(x_k)}{f'(x_k)}\,.$$

In Table 3 $\xi_{N,k}$ denotes the kth approximation with the initial approximation $\xi_{N,0}$ and the initial interval (a',b') in (4.3). Referring to the results in Table 2, one can see that the convergence rate was considerably improved by the hybrid method based on the presented method with small number of integration points N. On the other hand, though the hybrid methods associated with the Newton method and the Ostrowski method bring out excellent results for the equation $f_6(x)=0$, they require sufficiently large N in order to obtain convergent approximations for some improper equation such as $f_7(x)=0$.

Table 3. Numerical results of the hybrid iteration methods using the initial guess $\xi_{N,0}$ in (4.2) for each number of integration points $N=4,32,64$.

Examples	N	presented method		Newton method		Ostrowski method	
		$\lvert f_i(\xi_{N,k})\rvert$	NFE	$\lvert f_i(\xi_{N,k})\rvert$	NFE	$\lvert f_i(\xi_{N,k})\rvert$	NFE
$f_6(x)=0$	4	$(-25)_9$	18	$(-33)_8$	24	$(-50)_4$	16
	32	$(-32)_5$	10	$(-26)_5$	15	$(-61)_3$	12
	64	$(-32)_5$	10	$(-36)_5$	15	$(-20)_2$	8
$f_7(x)=0$	4	$(-25)_{11}$	22	div.	–	div.	–
	32	$(-31)_7$	14	div.	–	div.	–
	64	$(-36)_6$	12	$(-34)_6$	18	$(-37)_3$	12

In addition, we consider the test functions given in [2] for which Newton method is not appropriate as below.

Example 8. $f_8(x)=e^{1-x}-1\,,\quad 0\le x\le 7$

for which Newton method requires very many iterations to approach the root $p = 1$ when the initial approximation x_0 is far from 1.

Example 9. $f_9(x) = xe^{-x}, \quad -1 \le x \le 2$

for which the iterates obtained by Newton method, with any initial approximation $x_0 > 1$, move away from the root $p = 0$.

Numerical results for these examples are given in Table 4, which shows that the presented method can be a good alternative to Newton method in these cases as well.

Table 4. Numerical results for the examples for which Newton method is not appropriate.

Examples	presented method			Newton method		
	x_k	$\|f_i(x_k)\|$	NFE	x_k	$\|f_i(x_k)\|$	NFE
$f_8(x) = 0$	1.	$(-22)_6$	12	1.	$(-31)_{403}$	1209
$f_9(x) = 0$	-2.1×10^{-21}	$(-21)_6$	12	50.6836	$(-21)_{45}$	135

5. Conclusions

In this paper, for solving a nonlinear equation $f(x) = 0$, we have developed a new simple iterative method given by (2.5) which has the following properties:

i. Problem of choosing an appropriate initial approximation need not be concerned.
ii. Any derivatives of $f(x)$ are not needed.
iii. The hybrid method based on the presented method, with the initial guess $\xi_{N,0}$ in (4.2) and the initial interval (a', b') in (4.3), can highly improve the convergence rate with small number of additional function evaluations even if the equation is so pathological such as $f_7(x) = 0$.
iv. The convergence order is 2 which is equal to that of the Newton method. In practice, for most numerical examples the presented method is better (in the sense of the advantages i.–iii.) than the Newton method.

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Beong In Yun
Department of Informatics and Statistics
Kunsan National University
Gunsanr 573-701, Korea
E-mail address: biyun@kunsan.ac.kr

Miodrag S. Petković
Department of Mathematics
Faculty of Electronic Engineering
University of Niš
18000 Niš, Serbia
E-mail address: msp@eunet.rs