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On the new fourth-order methods for the simultaneous approximation of polynomial zeros

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ABSTRACT

A new iterative method of the fourth-order for the simultaneous determination of polynomial zeros is proposed. This method is based on a suitable zero-relation derived from the fourth-order method for a single zero belonging to the Schröder basic sequence. One of the most important problems in solving polynomial equations, the construction of initial conditions that enable both guaranteed and fast convergence, is studied in detail for the proposed method. These conditions are computationally verifiable since they depend only on initial approximations, the polynomial coefficients and the polynomial degree, which is of practical importance. The construction of improved methods in ordinary complex arithmetic and complex circular arithmetic is discussed. Finally, numerical examples and the comparison with existing fourth-order methods are given.

1. Introduction

The problem of solving polynomial equations ranks among the most significant in the theory and practice, not only of applied mathematics but also of many branches of engineering sciences, physics, chemistry, computer science, control theory, digital signal processing, bioscience, finance, and so on. Various methods for solving this challenging problem have been developed, such as methods of search and exclusion, methods based on fixed point relations, companion matrix methods, methods based on rational approximation, globally convergent algorithms that are applied interactively, and so on. Iterative methods for the simultaneous approximation of zeros of algebraic polynomials based on fixed point relations are frequently used powerful tool for solving polynomial equations; see, e.g. [1–3]. Extensive list of references related to zero-finding methods can be found there. Important practical interest in simultaneous methods has grown with parallel implementation of this class of methods since they run in several identical versions.

Let $P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n (n \geq 3)$ be a monic polynomial of order n with (real or complex) simple zeros, and let $z_1, \ldots, z_n$ be some approximations to the zeros $\zeta_1, \ldots, \zeta_n$ of P. The aim of this paper is to present a new iterative method for the simultaneous computation of polynomial zeros and to study its convergence properties. Defining $u(z) = f(z)/f'(z)$ and $A_k(z) = f^{(k)}(z)/(k!f'(z))$ $(k = 2, 3 \ldots)$, the construction of the proposed method relies on Schröder's iterative method of the fourth-order

$$\varphi_4(z) = z - u(z) - u(z)^2 A_2(z) - u(z)^3 (2A_2(z)^2 - A_3(z))$$

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and the substitution of the coefficients A_2 and A_3 by appropriate sums. In this way we derive the zero-relation

$$\zeta_i = z_i - u(z_i) - \frac{u(z_i)^2}{2(1-u(z_i)\Sigma_{1,i})^2}\left(\frac{P''(z_i)}{P'(z_i)} - u(z_i)(\Sigma_{1,i}^2 - \Sigma_{2,i})\right) \quad (i \in \boldsymbol{I}_n), \tag{1}$$

where $\boldsymbol{I}_n := \{1, \ldots, n\}$ is the index set and

$$\Sigma_{q,i} = \sum_{j \in \boldsymbol{I}_n \setminus i} \frac{1}{(z_i - \zeta_j)^q} \quad (q = 1, 2).$$

The zero-relation (1) is the base for the construction of simultaneous method (Section 2), whose fourth-order of convergence is proved in Section 3 assuming that initial approximations are reasonably close to the desired zeros.

Section 4 is devoted to another important task in the theory of iterative processes concerned with the construction of initial *computationally verifiable* conditions that provide the guaranteed convergence of the proposed method (18). In this manner, the characterization of "reasonably close approximations" is precisely established. These initial conditions are stated in the light of Smale's "point estimation theory" (see [3–5]) in the form

$$\max_{1\le i\le n} \frac{P(z_i^{(0)})}{\prod_{j\in \boldsymbol{I}_n\setminus i}(z_i^{(0)} - z_j^{(0)})} < \frac{1}{3n+1} \min_{\substack{1\le i,j\le n\\ i\ne j}} |z_i^{(0)} - z_j^{(0)}|$$

and depend only on attainable data—polynomial coefficients, its degree and initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ to the zeros. This is of great importance in practice since these conditions are computationally verifiable.

In Section 5, we point to some advantages of the presented zero-relation and the possibility of construction of improved methods in ordinary complex arithmetic and complex circular arithmetic. In particular, we construct two simultaneous methods with the accelerated convergence. Their order of convergence is increased from 4 to 5 and 6 using suitable corrections without additional calculations. In this manner, the computational efficiency of these accelerated methods is considerably improved.

Numerical results and the comparison with several existing fourth-order methods are given in Section 6.

2. Derivation of the fourth-order method

Let f be a real or complex function and let us define

$$u(z) = \frac{f(z)}{f'(z)}, \qquad A_k(z) = \frac{f^{(k)}(z)}{k!f'(z)} \quad (k = 2, 3 \ldots).$$

The first few members of Schröder's basic sequence $\{\varphi_k\}$ of order k are given by (omitting argument z on the right side)

$$\begin{aligned} \varphi_2(z) &= z - u, \\ \varphi_3(z) &= z - u - A_2, \\ \varphi_4(z) &= z - u - u^2A_2 - u^3(2A_2^2 - A_3) \end{aligned} \tag{2}$$

(see [6, p. 84]). Iterative function φ_4 of the fourth-order, given by (2), can be written in the form

$$\varphi_4(z) = z - u - u^2A_2(1 + 2uA_2) + A_3u^3. \tag{3}$$

In particular, let $f \equiv P$ be a monic polynomial of degree n having simple real or complex zeros $\zeta_1, \ldots, \zeta_n$, that is,

$$P(z) = \prod_{j=1}^{n}(z - \zeta_j). \tag{4}$$

Introduce the abbreviations:

$$A_{k,i} = A_k(z_i), \quad u = u(z) = \frac{P(z)}{P'(z)}, \qquad u_i = u(z_i),$$

$$\Sigma_{k,i}(z) = \sum_{j\in \boldsymbol{I}_n\setminus i} \frac{1}{(z-\zeta_j)^k}, \quad \Sigma_{k,i} = \Sigma_{k,i}(z_i)\ (k = 1, 2).$$

If two real or complex numbers w and z have moduli of the same order, that is, $|w| = O(|z|)$, we will write $w = O_M(z)$.

Using the logarithmic derivation, we find from (4)

$$\frac{\mathrm{d}}{\mathrm{d}z}\log P(z) = \frac{P'(z)}{P(z)} = \sum_{j=1}^{n} \frac{1}{z - \zeta_j}. \tag{5}$$

Let us rewrite (5) in the form

$$P'(z) = P(z)\sum_{j=1}^{n}\frac{1}{z-\zeta_j},$$

and find the second derivative,

$$P''(z) = P'(z)\sum_{j=1}^{n}\frac{1}{z-\zeta_j} - P(z)\sum_{j=1}^{n}\frac{1}{(z-\zeta_j)^2}.$$

Using the last relation and (5), we find

$$\begin{aligned}
2A_2(z) &= \frac{P''(z)}{P'(z)} = \sum_{j=1}^{n}\frac{1}{z-\zeta_j} - \frac{P(z)}{P'(z)}\sum_{j=1}^{n}\frac{1}{(z-\zeta_j)^2}\\
&= \sum_{j=1}^{n}\frac{1}{z-\zeta_j} - \left(\sum_{j=1}^{n}\frac{1}{z-\zeta_j}\right)^{-1}\sum_{j=1}^{n}\frac{1}{(z-\zeta_j)^2}\\
&= \left(\frac{1}{z-\zeta_i} + \sum_{j\in \mathbf{I}_n\setminus i}\frac{1}{z-\zeta_j}\right) - \left(\frac{1}{z-\zeta_i} + \sum_{j\in \mathbf{I}_n\setminus i}\frac{1}{z-\zeta_j}\right)^{-1}\left(\frac{1}{(z-\zeta_i)^2} + \sum_{j\in \mathbf{I}_n\setminus i}\frac{1}{(z-\zeta_j)^2}\right).
\end{aligned}$$

Let $z = z_i$ be a sufficiently close approximation to the zero ζ_i, and let $\varepsilon_i = z_i - \zeta_i$. From the last relation it follows

$$\begin{aligned}
2A_2(z_i) &= 2A_{2,i} = \frac{1}{\varepsilon_i} + \Sigma_{1,i} - \frac{1/\varepsilon_i^2 + \Sigma_{2,i}}{1/\varepsilon_i + \Sigma_{1,i}} = \frac{(1/\varepsilon_i + \Sigma_{1,i})^2 - 1/\varepsilon_i^2 - \Sigma_{2,i}}{1/\varepsilon_i + \Sigma_{1,i}}\\
&= \frac{2\Sigma_{1,i} + \varepsilon_i\Sigma_{1,i}^2 - \varepsilon_i\Sigma_{2,i}}{1+\varepsilon_i\Sigma_{1,i}} = 2\Sigma_{1,i} + O_M(\varepsilon_i).
\end{aligned} \tag{6}$$

Since

$$u(z_i) = \frac{P(z_i)}{P'(z_i)} = \left(\frac{1}{\varepsilon_i} + \Sigma_{1,i}\right)^{-1} = \frac{\varepsilon_i}{1+\varepsilon_i\Sigma_{1,i}} = O_M(\varepsilon_i),$$

from (6) we find

$$\Sigma_{1,i} = A_{2,i} + O_M(u_i) = A_{2,i} + c_1u_i, \tag{7}$$

where c_1 is a constant. After elementary but tedious calculations, in a similar way we obtain

$$\Sigma_{2,i} = A_{2,i}^2 - 2A_{3,i} + O_M(u_i) = A_{2,i}^2 - 2A_{3,i} + c_2u_i, \tag{8}$$

where c_2 is a constant. From (7) and (8) we find

$$A_{2,i} = \Sigma_{1,i} + c_1'u_i \tag{9}$$

$$A_{3,i} = \frac{A_{2,i}^2 - \Sigma_{2,i}}{2} + c_2'u_i, \tag{10}$$

where c_1' and c_2' are constants.

By (9) and (10), it follows from (3) for $z = z_i$

$$\begin{aligned}
\varphi_4(z_i) &= z_i - u_i - u_i^2A_{2,i}(1+2u_i(\Sigma_{1,i}+c_1'u_i)) + u_i^3\left(\frac{A_{2,i}^2-\Sigma_{2,i}}{2} + c_2'u_i\right)\\
&= z_i - u_i - u_i^2A_{2,i}(1+2u_i(\Sigma_{1,i}+c_1'u_i)) + u_i^3\left(\frac{(\Sigma_{1,i}+c_1'u_i)^2-\Sigma_{2,i}}{2} + c_2'u_i\right)\\
&= z_i - u_i - u_i^2A_{2,i}(1+2u_i\Sigma_{1,i}) + \frac{u_i^3}{2}(\Sigma_{1,i}^2 - \Sigma_{2,i}) + O_M(u_i^4)\\
&= z_i - u_i - u_i^2A_{2,i}(1+2u_i\Sigma_{1,i}) + \frac{u_i^3}{2}(\Sigma_{1,i}^2 - \Sigma_{2,i})(1+2u_i\Sigma_{1,i}) - \frac{u_i^3}{2}(\Sigma_{1,i}^2 - \Sigma_{2,i})\cdot 2u_i\Sigma_{1,i} + O_M(u_i^4),
\end{aligned}$$

whence

$$\varphi_4(z_i) = z_i - u_i - (1+2u_i\Sigma_{1,i})\left(u_i^2A_{2,i} - \frac{u_i^3}{2}(\Sigma_{1,i}^2 - \Sigma_{2,i})\right) + O_M(u_i^4). \tag{11}$$

Finally, using the development

$$\frac{1}{(1-u_i\Sigma_{1,i})^2} = 1 + 2u_i\Sigma_{1,i} + O_M(u_i^2)$$

in (11), we obtain

$$\varphi_4(z_i) \approx \widetilde{\varphi}_4(z_i) := z_i - u_i - \frac{u_i^2 A_{2,i}}{(1-u_i\Sigma_{1,i})^2} + \frac{u_i^3}{2(1-u_i\Sigma_{1,i})^2}(\Sigma_{1,i}^2 - \Sigma_{2,i}) \quad (i \in \boldsymbol{I}_n), \tag{12}$$

where we have neglected the terms next to u_i^k, $k \geq 4$.

Differentiating (5) we find

$$\frac{P'(z)^2 - P''(z)P(z)}{P(z)^2} = \sum_{j=1}^{n} \frac{1}{(z-\zeta_j)^2}. \tag{13}$$

From (5) and (13) we obtain for $z = z_i$

$$\Sigma_{1,i} = \frac{1}{u_i} - \frac{1}{\varepsilon_i}, \qquad 1 - u_i\Sigma_{1,i} = \frac{u_i}{\varepsilon_i}, \qquad \Sigma_{2,i} = \frac{1}{u_i^2} - \frac{P''(z_i)}{P(z_i)} - \frac{1}{\varepsilon_i^2}. \tag{14}$$

Substituting (14) in (13) yields

$$\begin{aligned}
\widetilde{\varphi}_4(z_i) &= z_i - u_i - \frac{u_i^2 A_{2,i}}{(u_i/\varepsilon_i)^2} + \frac{u_i^3(\Sigma_{1,i}^2 - \Sigma_{2,i})}{2(u_i/\varepsilon_i)^2} \\
&= z_i - u_i - \varepsilon_i^2 A_{2,i} + \frac{\varepsilon_i^2 u_i(\Sigma_{1,i}^2 - \Sigma_{2,i})}{2} \\
&= z_i - u_i - \frac{\varepsilon_i^2 P''(z_i)}{2P'(z_i)} + \frac{\varepsilon_i^2 P(z_i)}{2P'(z_i)}\left[\left(\frac{1}{u_i} - \frac{1}{\varepsilon_i}\right)^2 - \left(\frac{1}{u_i^2} - \frac{P''(z_i)}{P(z_i)} - \frac{1}{\varepsilon_i^2}\right)\right] \\
&= z_i - u_i - \frac{\varepsilon_i^2 P''(z_i)}{2P'(z_i)} + \frac{\varepsilon_i^2 P(z_i)}{2P'(z_i)}\left(\frac{2}{\varepsilon_i^2} - \frac{2}{u_i\varepsilon_i} + \frac{P''(z_i)}{P(z_i)}\right) \\
&= z_i - \varepsilon_i = z_i - (z_i - \zeta_i) = \zeta_i.
\end{aligned}$$

In this way we have proved that (12) defines a zero-relation, that is,

$$\zeta_i = z_i - u_i - \frac{u_i^2}{2(1-u_i\Sigma_{1,i})^2}\left(\frac{P''(z_i)}{P'(z_i)} - u_i(\Sigma_{1,i}^2 - \Sigma_{2,i})\right) \quad (i \in \boldsymbol{I}_n), \tag{15}$$

or in the form

$$\zeta_i = z_i - u_i - \frac{u_i^2}{2\left(1 - u_i\sum_{j\in \boldsymbol{I}_n\setminus i}\frac{1}{z_i-\zeta_j}\right)^2}\left[\frac{P''(z_i)}{P'(z_i)} - u_i\left(\left(\sum_{j\in \boldsymbol{I}_n\setminus i}\frac{1}{z_i-\zeta_j}\right)^2 - \sum_{j\in \boldsymbol{I}_n\setminus i}\frac{1}{(z_i-\zeta_j)^2}\right)\right] \quad (i \in \boldsymbol{I}_n). \tag{16}$$

Remark 1. The zero-relation (16) can be also derived using other methods, but the presented derivation is quite natural, originated from the Schröder method (2) of the fourth-order.

The relation (16) is suitable for the construction of simultaneous methods for finding real or complex simple zeros of a polynomial in complex arithmetic and the inclusion of simple zeros in circular complex interval arithmetic; see [7] for a general approach to the construction of simultaneous methods.

Introduce the abbreviations

$$\delta_{q,i} = \frac{P^{(q)}(z_i)}{P(z_i)}, \qquad S_{q,i} = \sum_{j\in \boldsymbol{I}_n\setminus i}\frac{1}{(z_i-z_j)^q} \quad (q = 1, 2).$$

Instead of $\sum_{j\in \boldsymbol{I}_n\setminus i}$ and $\prod_{j\in \boldsymbol{I}_n\setminus i}$ we will write sometimes $\sum_{j\neq i}$ and $\prod_{j\neq i}$.

Substituting the zeros $\zeta_1, \ldots, \zeta_n$ by their approximations $z_1, \ldots, z_n$ in (16), the following iterative method for the simultaneous determination of simple zeros of the polynomial P is obtained:

$$\hat{z}_i = z_i - u_i - \frac{u_i^2\left(\frac{P''(z_i)}{P'(z_i)} - u_i\left(S_{1,i}^2 - S_{2,i}\right)\right)}{2\left(1 - u_iS_{1,i}\right)^2}, \tag{17}$$

where

$$u_i = \frac{P(z_i)}{P'(z_i)} = \frac{1}{\delta_{1,i}}.$$

Here $\hat{z}_i$ denotes the approximation in the next iteration.

Assume that initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ to the zeros $\zeta_1, \ldots, \zeta_n$ of P have been found. Then the following new iterative method is obtained from (17)

$$z_i^{(m+1)} = z_i^{(m)} - u_i^{(m)} - \frac{\left(u_i^{(m)}\right)^2 \left(\frac{P''(z_i^{(m)})}{P'(z_i^{(m)})} - u_i^{(m)}\left(\left(S_{1,i}^{(m)}\right)^2 - S_{2,i}^{(m)}\right)\right)}{2\left(1 - u_i^{(m)} S_{1,i}^{(m)}\right)^2} \quad (m = 0, 1, \ldots) \tag{18}$$

for $i = 1, \ldots, n$. Here $u_i^{(m)}$ and $S_{q,i}^{(m)}$ $(q = 1, 2)$ are related to the mth iterative step.

3. The order of convergence

The following theorem states that the order of convergence of the method (18) is four.

Theorem 1. *If the initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ are sufficiently close to the respective zeros $\zeta_1, \ldots, \zeta_n$ of P, then the order of convergence of the iterative method (18) is four.*

Proof. Starting from the factorization $P(z) = \prod_{j=1}^{n}(z - \zeta_j)$ and using the logarithmic derivative, from (5) and (14) we obtain

$$\delta_{1,i} = \sum_{j=1}^{n} \frac{1}{z_i - \zeta_j} = \frac{1}{\varepsilon_i} + \Sigma_{1,i}, \qquad \delta_{1,i}^2 - \delta_{2,i} = \sum_{j=1}^{n} \frac{1}{(z_i - \zeta_j)^2} = \frac{1}{\varepsilon_i^2} + \Sigma_{2,i}, \tag{19}$$

where $\varepsilon_i = z_i - \zeta_i$.

Let us introduce the notations

$$G_{ij} = \frac{1}{(z_i - \zeta_j)(z_i - z_j)}, \qquad H_{ij} = \frac{(2z_i - z_j - \zeta_j)}{(z_i - \zeta_j)^2 (z_i - z_j)^2}. \tag{20}$$

Using (19) and (20), we find

$$\delta_{1,i} - S_{1,i} = \frac{1}{\varepsilon_i}\left(1 - \varepsilon_i \sum_{j \in \boldsymbol{I}_n \setminus i} G_{ij}\varepsilon_j\right), \qquad \delta_{1,i}^2 - \delta_{2,i} - S_{2,i} = \frac{1}{\varepsilon_i^2}\left(1 - \varepsilon_i^2 \sum_{j \in \boldsymbol{I}_n \setminus i} H_{ij}\varepsilon_j\right). \tag{21}$$

The iterative formula (17) can be represented as

$$\hat{z}_i = z_i - \frac{1}{\delta_{1,i} - S_{1,i}}\left(1 + \frac{\delta_{2,i} - \delta_{1,i}^2 + S_{2,i} + (\delta_{1,i} - S_{1,i})^2}{2\delta_{1,i}(\delta_{1,i} - S_{1,i})}\right). \tag{22}$$

Starting from (22) and using (21) we obtain

$$\hat{\varepsilon}_i = \hat{z}_i - \zeta_i = \varepsilon_i - \frac{\varepsilon_i}{\left(1 - \varepsilon_i \sum_{j \neq i} G_{ij}\varepsilon_j\right)}\left[1 + \frac{\varepsilon_i^2 \sum_{j \neq i} H_{ij}\varepsilon_j - 1 + \left(1 - \varepsilon_i \sum_{j \neq i} G_{ij}\varepsilon_j\right)^2}{2(1 + \varepsilon_i \Sigma_{1,i})\left(1 - \varepsilon_i \sum_{j \neq i} G_{ij}\varepsilon_j\right)}\right]$$

$$= \frac{\varepsilon_i^3\left(\left(\sum_{j \neq i} G_{ij}\varepsilon_j\right)^2 - \sum_{j \neq i} H_{ij}\varepsilon_j + 2\varepsilon_i \Sigma_{1,i}\left(\sum_{j \neq i} G_{ij}\varepsilon_j\right)^2 - 2\Sigma_{1,i}\sum_{j \neq i} G_{ij}\varepsilon_j\right)}{2(1 + \varepsilon_i \Sigma_{1,i})\left(1 - \varepsilon_i \sum_{j \neq i} G_{ij}\varepsilon_j\right)^2}.$$

Let $|\varepsilon| = \max_{1 \le j \le n} |\varepsilon_j|$ and let the absolute values of all errors ε_j $(j = 1, \ldots, n)$ be of the same order, that is, $|\varepsilon_j| = O(|\varepsilon|)$. The quantities G_{ij} and H_{ij} are bounded, and the denominator of the last expression is also bounded and tends to 2 when $|\varepsilon| \to 0$. According to these facts, from the last relation we have

$$|\hat{\varepsilon}| = \max_{1 \le j \le n} |\hat{\varepsilon}_j| = O\left(|\varepsilon|^4\right),$$

which completes the proof of Theorem 1. □

4. Convergence analysis—guaranteed convergence

Theorem 1 is proved by a standard technique assuming that initial approximations are "reasonably close" to the wanted zeros, without any available data which would characterize this closeness. Such a convergence technique is rather of theoretical interest and we will overcome this shortcoming in this section by presenting the convergence analysis of the method (18) using the approach based on Smale's point estimation theory [5]. This approach states computationally verifiable initial convergence conditions that guarantee the convergence of the considered methods so that it is regarded as an advance in the theory of iterative processes. As mentioned in Introduction, in the case of algebraic polynomials $P(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1}z + a_n$ it is assumed that initial conditions depend only on the polynomial coefficients $a_1, \ldots, a_n$, initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ and the polynomial degree n. Certainly, this is of great importance in practice since these conditions are computationally verifiable. More details on the point estimation theory concerning iterative methods for the simultaneous determination of polynomial zeros can be found in [8–13], and the references cited therein.

For distinct complex numbers $z_1, \ldots, z_n$, let us define

$$W_i = \frac{P(z_i)}{\prod\limits_{j\in \boldsymbol{I}_n\setminus i}(z_i - z_j)}, \qquad w = \max_{1\le j\le n} |W_j|, \qquad d = \min_{\substack{1\le i,j\le n\\ j\ne i}} |z_i - z_j|.$$

The following assertion was proved in [3], where parametric notation $\{c; r\} := \{z : |z-c| \le r\}$ denotes the disk with center c and radius r.

Theorem 2. *Let $z_1, \ldots, z_n$ be distinct numbers satisfying the inequality $w < c_n d,\ c_n < 1/(2n)$. Then the disks*

$$D_1 := \left\{z_1; \frac{|W_1|}{1-nc_n}\right\}, \ldots, D_n := \left\{z_n; \frac{|W_n|}{1-nc_n}\right\}$$

are mutually disjoint and each of them contains one and only one zero of the polynomial P, that is

$$\zeta_i \in \left\{z_i; \frac{|W_i|}{1-nc_n}\right\} \quad (i \in \boldsymbol{I}_n). \tag{23}$$

In what follows we suppose that the following condition

$$w < c_n d, \quad c_n = \frac{1}{3n+1}, \tag{24}$$

is satisfied. Since $c_n = 1/(3n+1) < 1/(2n)$, the assertions of Theorem 2 hold.

Starting from (23) we obtain

$$|\varepsilon_i| = |z_i - \zeta_i| < \frac{1}{1-nc_n}|W_i| < \frac{c_n}{1-nc_n} d = \frac{1}{2n+1} d = \gamma_n d, \tag{25}$$

where $\gamma_n = 1/(2n+1)$. Then

$$|z_i - \zeta_j| \ge |z_i - z_j| - |z_j - \zeta_j| > d - \frac{1}{2n+1} d = \frac{2n}{2n+1} d. \tag{26}$$

According to (25) and (26) we estimate

$$|\Sigma_{1,i}| \le \sum_{j\in \boldsymbol{I}_n\setminus i} \frac{1}{|z_i - \zeta_j|} < \frac{(n-1)(2n+1)}{2nd} \tag{27}$$

and

$$|1 + \varepsilon_i \Sigma_{1,i}| \ge 1 - |\varepsilon_i||\Sigma_{1,i}| \ge 1 - \frac{1}{(2n+1)} d \cdot \frac{(n-1)(2n+1)}{2nd} = 1 - \frac{n-1}{2n} = \frac{n+1}{2n} = \alpha_n. \tag{28}$$

Starting from (20) and taking into account (26), we estimate

$$\left|\sum_{j\in \boldsymbol{I}_n\setminus i} G_{ij}\varepsilon_j\right| \le \sum_{j\in \boldsymbol{I}_n\setminus i} \frac{|\varepsilon_j|}{|z_i - \zeta_j||z_i - z_j|} < \frac{(n-1)|\varepsilon_j|}{\frac{2n}{2n+1} d^2} < \frac{(n-1)\gamma_n}{\frac{2n}{2n+1} d} = \frac{n-1}{2nd} = \frac{a_n}{d} \tag{29}$$

and

$$\begin{aligned}\left|\sum_{j\in \boldsymbol{I}_n\setminus i} H_{ij}\varepsilon_j\right| &\le \sum_{j\in \boldsymbol{I}_n\setminus i} \frac{|\varepsilon_j|}{|z_i-\zeta_j||z_i-z_j|}\left(\frac{1}{|z_i-\zeta_j|}+\frac{1}{|z_i-z_j|}\right)\\ &\le \frac{(n-1)|\varepsilon_j|}{\frac{2n}{2n+1}d^2}\left(\frac{2n+1}{2nd}+\frac{1}{d}\right)\\ &\le \frac{(n-1)\gamma_n}{\frac{2n}{2n+1}d^2}\left(\frac{2n+1}{2n}+1\right)=\frac{(n-1)(4n+1)}{4n^2d^2}=\frac{b_n}{d^2},\end{aligned} \tag{30}$$

where

$$a_n=\frac{n-1}{2n}\quad\text{and}\quad b_n=\frac{(n-1)(4n+1)}{4n^2}.$$

Let us introduce

$$T_i=1+\frac{\delta_{2,i}-\delta_{1,i}^2+S_{2,i}+(\delta_{1,i}-S_{1,i})^2}{2\delta_{1,i}(\delta_{1,i}-S_{1,i})}.$$

Using (21), we find

$$T_i=1+\frac{\varepsilon_i\left(\varepsilon_i\left(\sum_{j\ne i}G_{ij}\varepsilon_j\right)^2+\varepsilon_i\sum_{j\ne i}H_{ij}\varepsilon_j-2\sum_{j\ne i}G_{ij}\varepsilon_j\right)}{2(1+\varepsilon_i\varSigma_{1,i})\left(1-\varepsilon_i\sum_{j\ne i}G_{ij}\varepsilon_j\right)}=1+t_i, \tag{31}$$

where

$$t_i=\frac{\varepsilon_i\left(\varepsilon_i\left(\sum_{j\ne i}G_{ij}\varepsilon_j\right)^2+\varepsilon_i\sum_{j\ne i}H_{ij}\varepsilon_j-2\sum_{j\ne i}G_{ij}\varepsilon_j\right)}{2(1+\varepsilon_i\varSigma_{1,i})\left(1-\varepsilon_i\sum_{j\ne i}G_{ij}\varepsilon_j\right)}.$$

Then the iterative formula (22) can be rewritten in the form

$$\hat{z}_i=z_i-C_i=z_i-T_i(\delta_{1,i}-S_{1,i})^{-1}. \tag{32}$$

Here C_i denotes the iterative correction which deals with the sums depending on the approximations $z_1,\ldots,z_n$.

Let

$$q_n:=\frac{\gamma_n\left(\gamma_n a_n^2+\gamma_n b_n+2a_n\right)}{2\alpha_n(1-\gamma_n a_n)}.$$

According to (25), (29) and (30) we estimate

$$\left|1-\varepsilon_i\sum_{j\in \boldsymbol{I}_n\setminus i}G_{ij}\varepsilon_j\right|\ge 1-|\varepsilon_i|\sum_{j\in \boldsymbol{I}_n\setminus i}|G_{ij}||\varepsilon_j|>1-\gamma_n d\cdot\frac{a_n}{d}=1-\gamma_n a_n=1-\frac{n-1}{2n(2n+1)}\ge\frac{20}{21} \tag{33}$$

and

$$\begin{aligned}|t_i|&\le\frac{|\varepsilon_i|\left(|\varepsilon_i|\left(\sum_{j\ne i}|G_{ij}||\varepsilon_j|\right)^2+|\varepsilon_i|\sum_{j\ne i}|H_{ij}||\varepsilon_j|+2\sum_{j\ne i}|G_{ij}||\varepsilon_j|\right)}{2\left|1+\varepsilon_i\varSigma_{1,i}\right|\left|1-\varepsilon_i\sum_{j\ne i}G_{ij}\varepsilon_j\right|}\\ &<\frac{\gamma_n d\left(\gamma_n d\cdot\frac{a_n^2}{d^2}+\gamma_n d\cdot\frac{b_n}{d^2}+\frac{2a_n}{d}\right)}{2\alpha_n\left(1-\gamma_n a_n\right)}=q_n,\end{aligned} \tag{34}$$

where the denominator of t_i is bounded according to (33). Under condition (24) we obtain

$$q_n = \frac{n(8n^2+n-9)}{2(n+1)(2n+1)(4n^2+n+1)} < 0.09.$$

Starting from (31) we find

$$|T_i| < 1 + |t_i| < 1 + q_n, \tag{35}$$

$$|T_i| > 1 - |t_i| > 1 - q_n. \tag{36}$$

The following lemma is concerned with some necessary bounds and estimates.

Lemma 1. *Let the inequality* (24) *hold. Then*

(i) $|\hat{z}_i - z_i| = |C_i| < 1.5|W_i|$;
(ii) $|\widehat{W}_i| < 0.3|W_i|$;
(iii) $\widehat{w} < c_n\hat{d}$, $c_n = 1/(3n+1)$.

Proof. For distinct points $z_1, \dots, z_n$, we use the Lagrangian interpolation to obtain the following representation of the polynomial P:

$$P(z) = \left[\sum_{j=1}^{n} \frac{W_j}{z - z_j} + 1\right] \prod_{j=1}^{n} (z - z_j), \qquad W_j = \frac{P(z_j)}{\prod_{k \in \boldsymbol{I}_n \setminus j} (z_j - z_k)}. \tag{37}$$

Putting $z = \hat{z}_i$ in (37), we obtain

$$P(\hat{z}_i) = \left(\frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j \neq i} \frac{W_j}{\hat{z}_i - z_j}\right) \prod_{j=1}^{n} (\hat{z}_i - z_j).$$

Hence, after dividing by $\prod_{j \in \boldsymbol{I}_n \setminus i} (\hat{z}_i - \hat{z}_j)$, we find

$$\widehat{W}_i = \frac{P(\hat{z}_i)}{\prod_{j \in \boldsymbol{I}_n \setminus i} (\hat{z}_i - \hat{z}_j)} = (\hat{z}_i - z_i)\left(\frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j \neq i} \frac{W_j}{\hat{z}_i - z_j}\right) \prod_{j \in \boldsymbol{I}_n \setminus i} \left(1 + \frac{\hat{z}_j - z_j}{\hat{z}_i - \hat{z}_j}\right). \tag{38}$$

In our consideration we will also use the identity (see [14])

$$\left(\delta_{1,i} - \sum_{j \in \boldsymbol{I}_n \setminus i} \frac{1}{z_i - z_j}\right) W_i = 1 + \sum_{j \in \boldsymbol{I}_n \setminus i} \frac{W_j}{z_i - z_j}. \tag{39}$$

Using (24) and the definition of the minimal distance d we obtain

$$\left|\left(\delta_{1,i} - S_{1,i}\right) W_i\right| \leq 1 + \sum_{j \in \boldsymbol{I}_n \setminus i} \frac{|W_j|}{|z_i - z_j|} < 1 + (n-1)c_n, \tag{40}$$

and

$$\left|\left(\delta_{1,i} - S_{1,i}\right) W_i\right| \geq 1 - \sum_{j \in \boldsymbol{I}_n \setminus i} \frac{|W_j|}{|z_i - z_j|} > 1 - (n-1)c_n. \tag{41}$$

Starting from the iterative formula (32) we find

$$\hat{z}_i - z_i = -C_i = -T_i(\delta_{1,i} - S_{1,i})^{-1}.$$

By (35) and (41) we estimate

$$|\hat{z}_i - z_i| = |C_i| = \frac{|T_i||W_i|}{|(\delta_{1,i} - S_{1,i})W_i|} < \frac{(1+q_n)|W_i|}{1-(n-1)c_n} = \frac{\lambda_n}{c_n}|W_i| < \lambda_n d, \tag{42}$$

where we put

$$\lambda_n = \frac{(1+q_n)c_n}{1-(n-1)c_n}.$$

The sequence λ_n/c_n has a complicated form and we used symbolic computation in the programming package *Mathematica* to find its upper bound,

$$\frac{\lambda_n}{c_n} < 1.5 \quad \text{for } n \geq 3.$$

Therefore, we have proved

$$|C_i| < 1.5|W_i|. \tag{43}$$

According to (42), we obtain

$$|\hat{z}_i - z_j| \geq |z_i - z_j| - |\hat{z}_i - z_i| > (1 - \lambda_n)d \tag{44}$$

and

$$|\hat{z}_i - \hat{z}_j| \geq |z_i - z_j| - |\hat{z}_i - z_i| - |\hat{z}_j - z_j| > (1 - 2\lambda_n)d. \tag{45}$$

The inequality (45) gives

$$\hat{d} > (1 - 2\lambda_n)d, \quad \text{that is, } \frac{d}{\hat{d}} < \frac{1}{1 - 2\lambda_n}. \tag{46}$$

We note that $\lambda_n < 0.14$ if (24) holds.

Starting from the iterative formula (32), by (31) and (39) we find

$$\frac{W_i}{\hat{z}_i - z_i} = -\frac{(\delta_{1,i} - S_{1,i})W_i}{T_i} = -\frac{1}{1 + t_i}\left(1 + \sum_{j \neq i} \frac{W_j}{z_i - z_j}\right). \tag{47}$$

To prove (ii) we use (47) and derive:

$$\begin{aligned}\frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j \in \boldsymbol{I}_n \setminus i} \frac{W_j}{\hat{z}_i - z_j} &= -\frac{1}{1 + t_i}\left(1 + \sum_{j \in \boldsymbol{I}_n \setminus i} \frac{W_j}{z_i - z_j}\right) + 1 + \sum_{j \in \boldsymbol{I}_n \setminus i} \frac{W_j}{\hat{z}_i - z_j} \\ &= \frac{-(\hat{z}_i - z_i) \sum_{j \in \boldsymbol{I}_n \setminus i} \frac{W_j}{(z_i - z_j)(\hat{z}_i - z_j)} + t_i\left(1 + \sum_{j \in \boldsymbol{I}_n \setminus i} \frac{W_j}{\hat{z}_i - z_j}\right)}{1 + t_i}.\end{aligned}$$

Hence, by (24), (34), (36), (42) and (44) we estimate

$$\begin{aligned}\left|\frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j \in \boldsymbol{I}_n \setminus i} \frac{W_j}{\hat{z}_i - z_j}\right| &\leq \frac{\lambda_n d \frac{(n-1)c_n d}{d(1-\lambda_n)d} + q_n\left(1 + \frac{(n-1)c_n d}{(1-\lambda_n)d}\right)}{1 - q_n} \\ &\leq \frac{\frac{(n-1)c_n\lambda_n}{1-\lambda_n} + q_n\left(1 + \frac{(n-1)c_n}{1-\lambda_n}\right)}{1 - q_n}.\end{aligned} \tag{48}$$

Using the bounds (42) and (45) we find

$$\left|\prod_{j \neq i}\left(1 + \frac{\hat{z}_j - z_j}{\hat{z}_i - \hat{z}_j}\right)\right| \leq \prod_{j \in \boldsymbol{I}_n \setminus i}\left(1 + \frac{|\hat{z}_j - z_j|}{|\hat{z}_i - \hat{z}_j|}\right) < \left(1 + \frac{\lambda_n}{1 - 2\lambda_n}\right)^{n-1}. \tag{49}$$

Starting from (38) and taking into account the inequalities (42), (48) and (49) we obtain

$$\begin{aligned}|\widehat{W}_i| &\leq |\hat{z}_i - z_i| \left|\frac{W_i}{\hat{z}_i - z_i} + 1 + \sum_{j \in \boldsymbol{I}_n \setminus i} \frac{W_j}{\hat{z}_i - z_j}\right| \left|\prod_{j \in \boldsymbol{I}_n \setminus i}\left(1 + \frac{\hat{z}_j - z_j}{\hat{z}_i - \hat{z}_j}\right)\right| \\ &< \frac{\lambda_n}{c_n}|W_i| \frac{\frac{(n-1)c_n\lambda_n}{1-\lambda_n} + q_n\left(1 + \frac{(n-1)c_n}{1-\lambda_n}\right)}{1 - q_n}\left(1 + \frac{\lambda_n}{1 - 2\lambda_n}\right)^{n-1} = f_n|W_i|,\end{aligned}$$

where we put

$$f_n = \frac{\lambda_n}{c_n} \cdot \frac{\frac{(n-1)c_n\lambda_n}{1-\lambda_n} + q_n\left(1 + \frac{(n-1)c_n}{1-\lambda_n}\right)}{1-q_n}\left(1 + \frac{\lambda_n}{1-2\lambda_n}\right)^{n-1}.$$

Using the programming package *Mathematica* we found that $f_n < 0.3$ for every $n \geq 3$. Hence,

$$|\widehat{W}_i| < f_n|W_i| < 0.3|W_i|. \tag{50}$$

Since $f_n < 0.3$ and $\lambda_n < 0.14$, it follows $\frac{f_n}{1-2\lambda_n} < 0.42 < 1$. Now we have by (46) and (50)

$$\widehat{w} < f_n w < f_n c_n d < \frac{f_n}{1-2\lambda_n} \cdot c_n \hat{d} < c_n \hat{d},$$

which proves (iii) of Lemma 1. □

To state the main result concerning the guaranteed convergence of the simultaneous methods (18), we first present a general theorem which can be applied to a wide class of simultaneous methods of the form

$$z_i^{(m+1)} = z_i^{(m)} - C_i\big(z_1^{(m)}, \ldots, z_n^{(m)}\big) \quad (i \in \boldsymbol{I}_n;\ m = 0, 1, \ldots), \tag{51}$$

where $z_1^{(m)}, \ldots, z_n^{(m)}$ are some distinct approximations to the simple zeros $\zeta_i, \ldots, \zeta_n$, respectively, obtained in the mth iterative step by the methods (51). As before, for simplicity, we will omit sometimes the iteration index m and denote quantities in the latter $(m+1)$th iteration by $\widehat{\ }$ ("hat").

Let $\Omega(\zeta_i)$ be a reasonably close neighborhood of the zero ζ_i $(i \in \boldsymbol{I}_n)$ and let us assume that the corrections C_i appearing in (51) can be represented as a ratio

$$C_i(z_1, \ldots, z_n) = \frac{P(z_i)}{F_i(z_1, \ldots, z_n)} \quad (i \in \boldsymbol{I}_n), \tag{52}$$

where the function $(z_1, \ldots, z_n) \mapsto F_i(z_1, \ldots, z_n)$ satisfies the following conditions for each $i \in \boldsymbol{I}_n$:

1° $F_i(\zeta_1, \ldots, \zeta_n) \neq 0$;
2° $F_i(z_1, \ldots, z_n) \neq 0$ for any $(z_1, \ldots, z_n) \in \Omega_1(\zeta_1) \times \cdots \times \Omega(\zeta_n)$;
3° $F_i(z_1, \ldots, z_n)$ is continuous in $\mathbb{C}^n$.

Define a real function $t \mapsto g(t)$ over the open interval $(0, 1)$ by

$$g(t) = \begin{cases} 1 + 2t, & 0 < t \leq \dfrac{1}{2} \\ \dfrac{1}{1-t}, & \dfrac{1}{2} < t < 1. \end{cases}$$

The following theorem, involving corrections C_i and the function g, plays the key role in the convergence analysis of iterative methods of the form (51).

Theorem 3. *Let the iterative method (51) have the correction term of the form (52) for which the conditions* 1°–3° *are satisfied, and let $z_1^{(0)}, \ldots, z_n^{(0)}$ be distinct initial approximations to the zeros of P. If there exists a real number $\beta \in (0, 1)$ such that the following two inequalities*

(i) $\left|C_i^{(m+1)}\right| \leq \beta \left|C_i^{(m)}\right| \quad (m = 0, 1, \ldots)$,

(ii) $\left|z_i^{(0)} - z_j^{(0)}\right| > g(\beta)\left(\left|C_i^{(0)}\right| + \left|C_j^{(0)}\right|\right) \quad (i \neq j,\ \ i, j \in \boldsymbol{I}_n)$,

hold, then the iterative method (51) is convergent.

See [15] for the proof.

According to (32), (51) and (52), in the case of the considered method (18) (written in the form (32)) the corrections C_i are given by

$$C_i = T_i(\delta_{1,i} - S_{1,i})^{-1} = \frac{P(z_i)}{F_i(z_1, \ldots, z_n)},$$

where

$$F_i(z_1,\ldots,z_n)=\frac{(\delta_{1,i}-S_{1,i})W_i\prod_{j\neq i}(z_i-z_j)}{T_i}\quad (i\in \boldsymbol{I}_n). \tag{53}$$

Now, we give the main result concerning the initial conditions that guarantee the convergence of the simultaneous method (18).

Theorem 4. *If the initial condition*

$$w^{(0)}<\frac{d^{(0)}}{3n+1} \tag{54}$$

holds, then the iterative method (18) *is convergent.*

Proof. According to the assertion (iii) of Lemma 1, the following implication

$$w<c_n d\Rightarrow \hat{w}<c_n\hat{d},\quad c_n=\frac{1}{3n+1}$$

is valid. In a similar way, we can prove by induction that the condition (54) implies the inequality $w^{(m)}<c_n d^{(m)}$ for each $m=1,2,\ldots$. Therefore, all assertions of Lemma 1 are valid for each $m=1,2,\ldots$ if the initial condition (54) holds. In particular, the inequalities

$$|W_i^{(m+1)}|<0.3|W_i^{(m)}| \tag{55}$$

and

$$|C_i^{(m)}|=|z_i^{(m+1)}-z_i^{(m)}|<1.5|W_i^{(m)}| \tag{56}$$

are valid for $i\in \boldsymbol{I}_n$ and $m=0,1,\ldots$.

From the iterative formula (32) we observe that the corrections $C_i^{(m)}$ are given by

$$C_i^{(m)}=T_i^{(m)}\left(\delta_{1,i}^{(m)}-S_{1,i}^{(m)}\right)^{-1}, \tag{57}$$

where the abbreviations $C_i^{(m)}, T_i^{(m)}, \delta_{1,i}^{(m)}, S_{1,i}^{(m)}$ are related to the mth iterative step.

We will prove that the iterative process (18) is well defined in each iteration if we show that the function $F_i(z_1,\ldots,z_n)=P(z_i)/C_i$ appearing in (53) does not vanish. In view of (35) and (41) and the definition of the minimal distance, we get from (54)

$$|F_i(z_1,\ldots,z_n)|=\left|\frac{(\delta_{1,i}-S_{1,i})W_i\prod_{j\neq i}(z_i-z_j)}{T_i}\right|>\frac{1-(n-1)c_n}{1+q_n}\cdot d^{n-1}>0.6d^{n-1}>0.$$

Hence, $F_i(z_1,\ldots,z_n)$ does not vanish.

The next step in our proof is to prove that the sequences $\{|C_i^{(m)}|\}$ $(i\in \boldsymbol{I}_n)$ are monotonically decreasing. Omitting the iteration index for simplicity, we obtain by using (31), (40), (43), (50) and (57)

$$\begin{aligned}|\widehat{C}_i| &< 1.5|\widehat{W}_i|<1.5\cdot 0.3|W_i|=0.45|W_i|=0.45|T_i(\delta_{1,i}-S_{1,i})^{-1}|\left|\frac{(\delta_{1,i}-S_{1,i})W_i}{T_i}\right|\\ &=0.45|C_i|\left|\frac{1}{1+t_i}\left(1+\sum_{j\neq i}\frac{W_j}{z_i-z_j}\right)\right|.\end{aligned}$$

Hence, by (36) and (40), we arrive at

$$|\widehat{C}_i|<0.45\frac{1+(n-1)c_n}{1-q_n}|C_i|.$$

It is easy to estimate

$$\frac{1+(n-1)c_n}{1-q_n}<1.35,$$

so that

$$|\widehat{C}_i|<0.45\cdot 1.35|C_i|<0.61|C_i|.$$

In this way we find that the constant β, which appears in Theorem 3, is equal to $\beta = 0.61$. Therefore, we have proved the inequality $|C_i^{(m+1)}| < 0.61|C_i^{(m)}|$ for each $i = 1, \ldots, n$ and $m = 0, 1, \ldots$.

The quantity $g(\beta)$ occurring in (ii) of Theorem 3 is equal to $g(0.61) = 1/(1 - 0.61) \le 2.57$. Using this fact we finally have to prove the disjunctivity of the inclusion disks

$$S_1 = \{z_1^{(0)}; g(0.61)|C_1^{(0)}|\}, \ldots, S_n = \{z_n^{(0)}; g(0.61)|C_n^{(0)}|\}$$

(assertion (ii) of Theorem 3). In regard to (56) we have $|C_i^{(0)}| < 1.5w^{(0)}$ for all $i = 1, \ldots, n$. Choosing the index $p \in \boldsymbol{I}_n$ so that

$$|C_p^{(0)}| = \max_{1 \le i \le n} |C_i^{(0)}|,$$

we obtain

$$d^{(0)} > (3n+1)w^{(0)} > \frac{1}{1.5}(3n+1)|C_p^{(0)}| \ge \frac{3n+1}{2 \cdot 1.5}\left(|C_i^{(0)}| + |C_j^{(0)}|\right) > g(0.61)\left(|C_i^{(0)}| + |C_j^{(0)}|\right)$$

since

$$\frac{3n+1}{2 \cdot 1.5} \ge 3.33 > 2.57 \ge g(0.61)$$

for all $n \ge 3$. This means that

$$|z_i^{(0)} - z_j^{(0)}| \ge d^{(0)} > g(0.61)\left(|C_i^{(0)}| + |C_j^{(0)}|\right) = \operatorname{rad} S_i + \operatorname{rad} S_j.$$

Hence, according to a simple geometric construction, it follows that the inclusion disks $S_1, \ldots, S_n$ are disjoint, which completes the proof of Theorem 4. □

Combining Theorems 1 and 4, we state the following theorem.

Theorem 5. *If the initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ satisfy the initial condition (54), then the iterative method (18) converges with the order of convergence four.*

5. Acceleration of convergence and other modifications

The significant advantages of the zero-relation (16) and the iterative method (18) result from the following nice properties.

(1) The method (18) has a suitable structure which allows significant acceleration of convergence with negligible number of additional basic operations. Obviously, the computational efficiency of these accelerated methods is significantly increased.

(2) Using convenient transformations, the iterative formula (18) can be modified to the form suitable for finding multiple zeros of polynomials,

$$\hat{z}_i = z_i - \mu_i u_i - \frac{\mu_i u_i \left(1 - \mu_i + u_i \mu_i \frac{P''(z_i)}{P'(z_i)} - u_i^2 \left(\widetilde{S}_{1,i}^2 - \mu_i \widetilde{S}_{2,i}\right)\right)}{2\left(1 - u_i \widetilde{S}_{1,i}\right)^2} \quad (i \in \boldsymbol{I}_\nu := \{1, \ldots, \nu\}, \nu \le n),$$

where $\mu_1, \ldots, \mu_\nu$ are the multiplicities of multiple zeros $\zeta_1, \ldots \zeta_\nu$ and

$$\widetilde{S}_{q,i} = \sum_{j \in \boldsymbol{I}_\nu \setminus i} \frac{\mu_j}{(z_i - z_j)^q} \quad (q = 1, 2).$$

(3) The zero-relation (16) is convenient for the construction of interval methods for the simultaneous inclusion of polynomial zeros in complex circular arithmetic. For example, if $Z_1, \ldots, Z_n$ are disks which contain the polynomial zeros, then the following inclusion method of the fourth-order is obtained

$$\hat{Z}_i = z_i - u_i - \frac{u_i^2}{2\left(1 - u_i \sum_{j \in \boldsymbol{I}_n \setminus i} \frac{1}{z_i - Z_j}\right)^2} \left[\frac{P''(z_i)}{P'(z_i)} - u_i\left(\left(\sum_{j \in \boldsymbol{I}_n \setminus i} \frac{1}{z_i - Z_j}\right)^2 - \sum_{j \in \boldsymbol{I}_n \setminus i} \frac{1}{(z_i - Z_j)^2}\right)\right] \quad (i \in \boldsymbol{I}_n),$$

where z_i is the center of the disk Z_i. Recall that the crucial advantage of inclusion methods consists of automatic determination of the upper error bounds given by radii of produced disks that contain the wanted zeros in each iteration. Moreover, using the approach presented in [16], the convergence of the above inclusion method can be increased to five and six without additional numerical operations. Let us emphasize that the interval arithmetic, as a powerful device in controlling rounding errors and the inclusion of exact results, becomes a composite part of new modern computer arithmetics; see [17].

The corresponding methods of the form (18) based on the properties (2) and (3) will be investigated in detail in the forthcoming research, together with detailed convergence properties of the accelerated methods mentioned in point (1). In this paper, we discuss the convergence speed of accelerated methods in short, including two numerical examples. Note that many simultaneous methods, including Weierstrass–Dochev's method [18,19] (also known as Durand–Kerner's method [20,21])

$$\hat{z}_i = z_i - \frac{P(z_i)}{\prod_{j\in \boldsymbol{I}_n\setminus i}(z_i - z_j)} = z_i - W_i, \tag{58}$$

do not possess the properties (1) and (2). Furthermore, the complex interval variant of Weierstrass–Dochev's method has a low computational efficiency; see [7, Ch. 6].

For simplicity, let us omit the iteration index m. Beside the vector of current approximations $\boldsymbol{z}^{(1)} = (z_1^{(1)}, \ldots, z_n^{(1)}) := (z_1, \ldots, z_n)$, we will also consider the following improved approximations $\boldsymbol{z}^{(k)} = (z_1^{(k)}, \ldots, z_n^{(k)})$ $(k = 2, 3)$, where

$$z_j^{(2)} = z_j - u_j = z_j - \frac{P(z_j)}{P'(z_j)} \quad \text{(Newton's approximations)},$$

$$z_j^{(3)} = z_j - h_j = z_j - \frac{P(z_j)}{P'(z_j) - \frac{P(z_j)P''(z_j)}{2P'(z_j)}} \quad \text{(Halley's approximations)}.$$

The Newton and Halley approximations occur in the classic iterative methods

$$\hat{z}_j = z_j - u_j = z_j - \frac{P(z_j)}{P'(z_j)} = z_j - \frac{1}{\delta_{1,j}} \quad \text{(Newton's method, order 2)},$$

$$\hat{z}_j = z_j - h_j = z_j - \frac{P(z_j)}{P'(z_j) - \frac{P(z_j)P''(z_j)}{2P'(z_j)}} = z_j - \frac{2\delta_{1,j}}{2\delta_{1,j}^2 - \delta_{2,j}} \quad \text{(Halley's method, order 3)}.$$

We emphasize that superscript indices now indicate the type of approximations and they should be strongly distinguished from the iteration index.

Let us define the sums

$$\left(S_{q,i}\right)_k = \sum_{j\in \boldsymbol{I}_n\setminus i} \frac{1}{\left(z_i - z_j^{(k)}\right)^q} \quad (q = 1, 2;\ k = 1, 2, 3),$$

where the index k points to the type of approximations $z_j^{(k)}$ $(k = 1, 2, 3)$. Then from the zero-relation (16) we construct the following iterative method:

$$\hat{z}_i = z_i - u_i - \frac{u_i^2\left[\frac{P''(z_i)}{P'(z_i)} - u_i\left(\left(S_{1,i}\right)_k^2 - \left(S_{2,i}\right)_k\right)\right]}{2\left(1 - u_i\left(S_{1,i}\right)_k\right)^2} \quad (k = 1, 2, 3). \tag{59}$$

For $k = 1$ (the use of current approximations $z_j^{(1)} = z_j$) the method (59) reduces to the fourth-order method without corrections (18). For $k = 2$ and $k = 3$ the iterative formula (59) defines two new simultaneous methods with corrections having accelerated convergence. The order of convergence of the method (59) is given in the following theorem.

Theorem 6. *If the initial approximations $z_1^{(0)}, \ldots, z_n^{(0)}$ are sufficiently close to the respective zeros $\zeta_1, \ldots, \zeta_n$ of P, then the order of convergence of the iterative method (59) is $k + 3$ $(k = 1, 2, 3)$.*

The proof is similar to that given in [10] and we omit it.

Let us note that the acceleration of convergence of the method (59) from 4 to 5 and 6 is attained using already calculated quantities. Therefore, computational efficiency of the accelerated methods (59) (for $k = 2, 3$) is considerably increased.

6. Numerical results

To demonstrate the convergence properties of the new methods (18) and (59), we have applied these methods to a number of polynomial equations. For comparison purpose, beside the new methods (18) and (59) we have also tested the following simultaneous methods of the fourth-order.

Modified Ehrlich's method [22]:

$$\hat{z}_i = z_i - \frac{1}{\frac{1}{u_i} - \sum\limits_{j\in \boldsymbol{I}_n\setminus i} \frac{1}{z_i - z_j + u_j}}, \quad u_i = \frac{P(z_i)}{P'(z_i)}. \tag{ME}$$

Halley-like method [23]:

$$\hat{z}_i = z_i - \frac{2\delta_{1,i}}{2\delta_{1,i}^2 - \delta_{2,i} - S_{2,i} - S_{1,i}^2}. \tag{HM}$$

Ostrowski-like method [24]:

$$\hat{z}_i = z_i - \frac{1}{\sqrt{\delta_{1,i}^2 - \delta_{2,i} - S_{2,i}}}. \tag{OM}$$

Modified Börsch–Supan method [25]:

$$\hat{z}_i = z_i - \frac{W_i}{1 - \sum\limits_{j\in \boldsymbol{I}_n\setminus i} \frac{W_j}{z_i - z_j + W_j}}, \quad W_i = \frac{P(z_i)}{\prod\limits_{j\in \boldsymbol{I}_n\setminus i} (z_i - z_j)}. \tag{MBS}$$

Kyurkchiev's method [26] (see, also, [27]):

$$\hat{z} = z_i - \frac{W_i}{1 + \sum\limits_{j\in \boldsymbol{I}_n\setminus i} \frac{W_j}{z_i - z_j} + W_i \sum\limits_{j\in \boldsymbol{I}_n\setminus i} \frac{W_j}{(z_i - z_j)^2}}. \tag{KM}$$

Double Weierstrass–Dochev's method [18,19]:

$$y_i = z_i - \frac{P(z_i)}{\prod\limits_{j\in \boldsymbol{I}_n\setminus i} (z_i - z_j)}, \qquad \hat{z}_i = y_i - \frac{P(y_i)}{\prod\limits_{j\in \boldsymbol{I}_n\setminus i} (y_i - y_j)}. \tag{DWD}$$

Let us note that the method (DWD) is obtained by applying Weierstrass–Dochev's method (58) applied two times. Considered as a two-point method, the method (DWD) has the order four. This artificial composition is made for comparison with the presented fourth-order methods.

In our numerical experiments, we have often used the fact that all zeros of a polynomial $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ $(a_0, a_n \neq 0)$ lie inside the annulus $\{z \in \mathbb{C} : r < |z| < R\}$, where r and R are calculated as

$$r = \frac{1}{2} \min_{1\le k\le n} \left| \frac{a_n}{a_{n-k}} \right|^{1/k}, \qquad R = 2 \max_{1\le k\le n} \left| \frac{a_k}{a_0} \right|^{1/k} \tag{60}$$

(see [28, Theorem 6.4b, Corollary 6.4k]). The measure of approximations produced in the iterative process is given by the norm

$$e^{(m)} = \left(\sum_{j=1}^{n} |z_j^{(m)} - \zeta_j|^2 \right)^{1/2} \quad (m = 0, 1, \ldots).$$

Example 1. The new method (18), the new methods with Newton's corrections (59)—N-cor (order 5) and Halley's corrections (59)—H-cor (order 6), and the listed methods (ME), (HM), (OM), (MBS), (KM) and (DWD) were applied for the simultaneous approximation to the zeros of the polynomial

$$\begin{aligned} P(z) = {} & z^{19} - 3z^{18} + 12z^{17} - 36z^{16} + 268z^{15} - 804z^{14} + 2784z^{13} - 8352z^{12} + 34\,710z^{11} \\ & - 104\,130z^{10} + 324\,696z^9 - 974\,088z^8 + 620\,972z^7 - 1\,862\,916z^6 - 2\,270\,592z^5 \\ & + 6\,811\,776z^4 - 28\,303\,951z^3 + 84\,911\,853z^2 - 25\,704\,900z + 77\,114\,700. \end{aligned}$$

The zeros of this polynomial are $\pm 1 \pm 2\mathrm{i}$, ± 2, $\pm \mathrm{i}$, $\pm 3 \pm 2\mathrm{i}$, $\pm 2 \pm 3\mathrm{i}$, $\pm 3\mathrm{i}$, 3. Initial approximations were taken to give $e^{(0)} = 0.69$.

The errors $e^{(m)}$ of approximations in the first three iterations are given in Table 1, where $A(-h)$ means $A \times 10^{-h}$.

Table 1
The errors of approximations in the first three iterations, Example 1.

Methods	$e^{(1)}$	$e^{(2)}$	$e^{(3)}$
New (18)	4.39(−3)	1.51(−11)	8.98(−45)
(ME)	5.03(−3)	2.97(−11)	4.01(−44)
(HM)	4.32(−3)	1.37(−11)	5.19(−45)
(OM)	1.58(−3)	1.40(−13)	6.24(−54)
(MBS)	3.16(−3)	7.05(−12)	2.90(−47)
(KM)	3.47(−3)	1.06(−11)	1.01(−45)
(DWD)	1.08(−2)	1.58(−9)	7.15(−37)
(59)–N-cor	1.96(−3)	4.85(−15)	4.15(−72)
(59)–H-cor	4.93(−4)	4.29(−21)	1.59(−122)

Table 2
The errors of approximations in the first three iterations, Example 2.

Methods	$e^{(1)}$	$e^{(2)}$	$e^{(3)}$
New (18)	2.09(−1)	1.02(−4)	6.59(−18)
(ME)	2.06(−1)	1.21(−4)	3.10(−17)
(HM)	2.30(−1)	1.65(−4)	9.61(−17)
(OM)	Diverges	–	–
(MBS)	1.27(−1)	1.69(−5)	6.82(−21)
(KM)	1.25(−1)	1.73(−5)	1.83(−20)
(DWD)	3.51(−1)	1.13(−3)	2.60(−13)
(59)—N-cor	1.30(−1)	4.43(−6)	2.21(−27)
(59)—H-cor	8.89(−2)	1.53(−8)	4.16(−49)

Example 2. In order to find the zeros of the polynomial

$$\begin{aligned} P(z) &= z^{20} + 12z^{19} + 80z^{18} + 360z^{17} + 1356z^{16} + 4512z^{15} + 13\,440z^{14} + 35\,520z^{13} + 84\,976z^{12} \\ &\quad + 192\,192z^{11} + 416\,000z^{10} + 574\,080z^{9} - 153\,024z^{8} - 3\,283\,968z^{7} - 8\,048\,640z^{6} \\ &\quad - 15\,452\,160z^{5} - 20\,317\,184z^{4} - 15\,925\,248z^{3} - 38\,010\,880z^{2} - 68\,812\,800z - 73\,728\,000 \end{aligned}$$

we applied the same methods as in Example 1. The zeros of this polynomial are $1 \pm \mathrm{i}$, $1 \pm 3\mathrm{i}$, $2 \pm 2\mathrm{i}$, ± 2, $\pm 2\mathrm{i}$, $-1 \pm \mathrm{i}$, $-1 \pm 3\mathrm{i}$, $-2 \pm 2\mathrm{i}$, $-3 \pm \mathrm{i}$, $-3 \pm 3\mathrm{i}$. The initial approximations were selected to give $e^{(0)} = 1.59$. The entries of the errors of approximations produced in the first three iterations are given in Table 2. The worse results compared with Example 1 are the consequence of crude initial approximations.

Example 3. The new method (18) was applied for finding the zeros of the monic polynomial P of degree 20 given by

$$\begin{aligned} P(x) &= x^{20} + (0.887 - 0.342\mathrm{i})x^{19} + (-0.569 + 0.909\mathrm{i})x^{18} + (0.109 + 0.855\mathrm{i})x^{17} \\ &\quad + (0.294 - 0.651\mathrm{i})x^{16} + (-0.087 + 0.948\mathrm{i})x^{15} + (-0.732 + 0.921\mathrm{i})x^{14} \\ &\quad + (0.801 - 0.573\mathrm{i})x^{13} + (0.506 - 0.713\mathrm{i})x^{12} + (-0.670 + 0.841\mathrm{i})x^{11} \\ &\quad + (-0.369 - 0.682\mathrm{i})x^{10} + (0.177 - 0.946\mathrm{i})x^{9} + (-0.115 + 0.577\mathrm{i})x^{8} \\ &\quad + (0.174 - 0.956\mathrm{i})x^{7} + (-0.018 - 0.438\mathrm{i})x^{6} + (0.738 + 0.645\mathrm{i})x^{5} \\ &\quad + (-0.655 - 0.618\mathrm{i})x^{4} + (0.123 - 0.088\mathrm{i})x^{3} + (0.773 + 0.965\mathrm{i})x^{2} \\ &\quad + (-0.757 + 0.109\mathrm{i})x + 0.223 - 0.439\mathrm{i}. \end{aligned}$$

The coefficients $a_k \in \mathbb{C}$ (except the leading unit coefficient) were chosen by the random generator as $\mathrm{Re}(a_k) = \texttt{random(x)}$, $\mathrm{Im}(a_k) = \texttt{random(x)}$, where $\texttt{random(x)} \in (-1, 1)$ and the random numbers are truncated up to three decimal digits.

Using (60) we find that all zeros of the above polynomial lie in the annulus $\{x : r = 0.3155 < |z| < 2.0711 = R\}$. We could start with initial approximations equidistantly spaced on the circle with radius $r_0 \in (0.3155, 2.0711)$, but we wanted to test the new method (18) in the case of very far initial approximations. For this reason, we chose initial approximations on the circle $|z| = 10$, determined in the following way:

$$z_\nu^{(0)} = r_0 \exp(\mathrm{i}\theta_\nu), \quad \mathrm{i} = \sqrt{-1},\ \theta_\nu = \frac{\pi}{n}\left(2\nu - \frac{3}{2}\right)\ (\nu = 1, \ldots, 20)$$

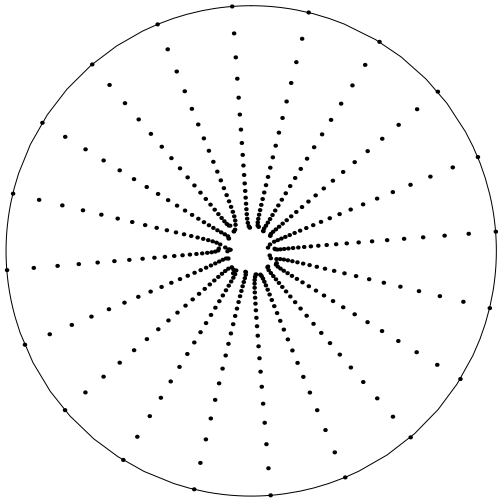

Fig. 1. The flow of the iterative process (18).

(the so-called Aberth's approximations; see [29]). We terminated the iterative process when the stopping criterion

$$\max_{1\le i\le 20} |P(z_i^{(m)})| < \tau = 10^{-12}$$

was satisfied.

The behavior of the iterative method (18) for the given polynomial is illustratively displayed in Fig. 1. The stopping criterion was satisfied after 23 iterations. At the beginning, the method converges linearly but almost straightforwardly toward the exact zeros, showing in several final iterations the fourth-order convergence. One can observe that approximations are radially distributed toward the aimed zeros.

According to the results shown in Tables 1 and 2 and a number of tested polynomials, we can conclude that the new method (18) is competitive with the existing simultaneous methods of the same order. However, its modification (59) with Newton's and Halley's corrections gives considerably better results.

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