On an iterative method for simultaneous inclusion of polynomial complex zeros

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ABSTRACT

Starting from disjoint discs which contain polynomial complex zeros, the iterative interval method of the third order for the simultaneous finding inclusive discs for complex zeros is formulated. The Lagrangean interpolation formula and complex circular arithmetic are used. The convergence theorem and the conditions for convergence are considered. The proposed method has been applied for solving an algebraic equation.

1. INTRODUCTION

The iterative interval methods for the simultaneous determination of polynomial complex zeros were the subject of many papers (see [7], [8], [9], [10], [11], [16], [17]). These methods are derived as interval versions of the known iterative processes in ordinary arithmetic.
The improvement of inclusive discs containing the exact polynomial zeros, in a sense of contraction, is done by iterative procedure using complex circular arithmetic.
D. Braess and K. P. Hadeler [4] applied Lagrangean interpolation formula for the simultaneous inclusion of the polynomial zeros by discs in a complex plane. The optimization of these discs leads to a special type of matrix eigenvalue problem. In section 2 of this paper a new simultaneous interval method is derived. This method also applies Lagrangean interpolation, but the contraction of inclusive discs is performed by iterative procedure in circular arithmetic.
The proposed interval process may be regarded as a version of classical result introduced by Weierstrass [19] (see, also, [1], [3], [5], [6], [13], [18]) with error bounds; it is proved that these bounds converge cubically to zero.
The application of the suggested simultaneous method for solving an algebraic equation is shown in section 3.

2. INTERVAL METHOD

Consider a polynomial of degree $n \geqslant 3$

$$P(z) = \prod_{j=1}^{n} (z - \xi_j)$$

with simple real or complex zeros $\xi_1,\ldots,\xi_n$. Suppose that disjoint discs $Z_i = \{\mathrm{mid}(Z_i); \mathrm{rad}\,(Z_i)\} = \{z_i; r_i\}$ with the center z_i and the radius r_i contain the zeros ξ_i $(i=1,\ldots,n)$. The polynomial P is identical with its Lagrangean interpolation polynomial for the points $z_1,\ldots,z_n$ and ∞, i.e.

$$P(z) = \sum_{j=1}^{n} \frac{Q(z)}{Q'(z_j)(z-z_j)} P(z_j) + Q(z),$$

where

$$Q(z) = (z-z_1)(z-z_2)\ldots(z-z_n).$$

Suppose that $P(z_i) \neq 0$ for each $i = 1,\ldots,n$. For any zero ξ_i $(i \in \{1,\ldots,n\})$ of P we have

$$\sum_{j=1}^{n} \frac{P(z_j)}{Q'(z_j)(\xi_i - z_j)} = -1$$

or

$$\frac{1}{\xi_i - z_i} \cdot \frac{P(z_i)}{Q'(z_i)} + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{P(z_j)}{Q'(z_j)(\xi_i - z_j)} = -1.$$

With the abbreviation

$$h_j = \frac{P(z_j)}{Q'(z_j)} = \frac{P(z_j)}{\prod_{\substack{k=1\\ k\neq j}}^{n} (z_j - z_k)}$$

we obtain

$$\xi_i = z_i - \frac{h_i}{1 - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{h_j}{z_j - \xi_i}} \quad (i=1,\ldots,n). \tag{2.1}$$

Since $\xi_i \in Z_i$ $(i=1,\ldots,n)$, on the basis of the inclusion monotonocity property from (2.1) it follows

$$\xi_i \in z_i - \frac{h_i}{1 - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{h_j}{z_j - Z_i}} \quad (i=1,\ldots,n). \tag{2.2}$$

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Accordingly, if the interval Z_i contains the zero ξ_i of the polynomial P, then the interval (in the form of a disc)

$$z_i - \frac{h_i}{1 - \sum\limits_{\substack{j=1\\ j\neq i}}^{n} \frac{h_j}{z_j - Z_i}}$$

also contains ξ_i (i=1,...,n).

Introduce the following notations

$$\rho = \min_{\substack{i,j\\ i\neq j}} \{|z| : z \in z_i - Z_j\} = \min_{\substack{i,j\\ i\neq j}} \{|z_i - z_j| - r_j\},$$

$$r = \max_{1\leqslant j\leqslant n} r_j,$$

$$H = \max_{1\leqslant j\leqslant n} |h_j|,$$

$$\eta = \frac{(n-1)rH}{\rho^2},$$

$$w_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{h_j}{z_j - z_i},$$

$$W_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{h_j}{z_j - Z_i}.$$

It is known (see, e.g. [2, Ch. 6]) that the disc $Z_1 = \{c_1; r_1\}$ contains the disc $Z_2 = \{c_2; r_2\}$, denoted by $Z_2 \subseteq Z_1$, if and only if

$$|c_1 - c_2| \leqslant r_1 - r_2. \tag{2.3}$$

Discs Z_1 and Z_2 are disjoint ($Z_1 \cap Z_2 = \phi$) if and only if

$$|c_1 - c_2| > r_1 + r_2. \tag{2.4}$$

By the properties of complex circular arithmetic we can prove the inequality

$$\left|\frac{1}{z_j - z_i} - \text{mid}\left(\frac{1}{z_j - Z_i}\right)\right| < \frac{r}{\rho^2} - \text{rad}\left(\frac{1}{z_j - Z_i}\right).$$

Hence, according to (2.3), it follows

$$\frac{1}{z_j - Z_i} \subset \left\{\frac{1}{z_j - z_i}; \frac{r}{\rho^2}\right\},$$

so that

$$\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{h_j}{z_j - Z_i} \subset \left\{\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{h_j}{z_j - z_i}; \frac{(n-1)rH}{\rho^2}\right\} = \{w_i; \eta\}. \tag{2.5}$$

Lemma 1

Under the condition

$$\rho > 3(n-1)r, \tag{2.6}$$

the inequality

$$H < \alpha r \tag{2.7}$$

holds, where $\alpha = e^{1/3} \cong 1.3956$.

Proof

The sequence $a(k) = (1 + \frac{1}{3k})^k$ is bounded and monotonically increasing so that

$$a(k) < \lim_{k\to+\infty} a(k) = e^{1/3}$$

for each $k \in N$. According to this, for each $j \in \{1,...,n\}$ we have the following estimate

$$|h_j| = \frac{|P(z_j)|}{|Q'(z_j)|} = |z_j - \xi_j| \prod_{\substack{i=1\\ i\neq j}}^{n} \left|\frac{z_j - \xi_i}{z_j - z_i}\right|$$

$$< r_j \prod_{\substack{i=1\\ i\neq j}}^{n} \frac{|z_j - z_i| + r_i}{|z_j - z_i|} < r\left(1 + \frac{r}{\rho}\right)^{n-1}$$

$$< r\left[1 + \frac{1}{3(n-1)}\right]^{n-1} < e^{1/3} r,$$

that is

$H < \alpha r$. □

Lemma 2

If (2.6) holds, then the discs $1 - W_i$ (i=1,...,n) do not contain the origin; more precisely, none of these discs intersects the disc centered at the origin with radius

$$\epsilon(n) = 1 - \frac{1}{3}\left[1 + \frac{1}{3(n-1)}\right]^n.$$

Proof

On the basic of the inclusion (2.5) it is sufficient to prove that for any $i \in \{1,...,n\}$ the disc $\{1 - w_i; \eta\}$ does not intersect the disc $\{0; \epsilon(n)\}$. This requirement is equivalent to the inequality

$$|1 - w_i| > \eta + \epsilon(n). \tag{2.8}$$

Since

$$|1 - w_i| = \left|1 - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{h_j}{z_j - z_i}\right| > 1 - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{|h_j|}{|z_j - z_i|} > 1 - \frac{(n-1)H}{\rho},$$

by virtue of (2.6), we have

$$\epsilon(n) = 1 - \frac{1}{3}\left[1 + \frac{1}{3(n-1)}\right]^n < 1 - (n-1)\frac{r}{\rho}\left(1 + \frac{r}{\rho}\right)^n$$

$$< 1 - \frac{(n-1)H(\rho + r)}{\rho^2}$$

$$= 1 - \frac{(n-1)H}{\rho} - \frac{(n-1)Hr}{\rho^2} < |1 - w_i| - \eta.$$

Therefore, the inequality (2.8) is valid. □

Remark 1

It can be shown that the sequence $\epsilon(n)$ is bounded and monotonically increasing so that for each $n \geqslant 3$ we have

$$\epsilon(3) \leqslant \epsilon(n) < \lim_{n\to+\infty} \epsilon(n) = 1 - \frac{e^{1/3}}{3} \cong 0.535.$$

Since $\epsilon(3) \cong 0.47$ it follows that none of the discs $1-W_i (i=1,\ldots,n)$ intersects the disc $\{0; 0.47\}$ for any $n \geqslant 3$.
Suppose that the disjoint discs $Z_i^{(0)} = \{z_i^{(0)}; r_i^{(0)}\}$, which contain the polynomial zeros ξ_i $(i=1,\ldots,n)$, are found. The relation (2.2) suggests a new interval method for the simultaneous finding of simple polynomial complex zeros with automatic errorbound.
Let $m=0,1,2,\ldots$ be the index of iteration, and let $\rho^{(m)}$, $r^{(m)}$, $h_j^{(m)}$, $H^{(m)}$, $\eta^{(m)}$, $w_i^{(m)}$, $W_i^{(m)}$ be notations, introduced above, in reference to the m-th iterative step. Further, let

$$\lambda^{(m)} = \frac{r^{(m)}}{\rho^{(m)}}, \quad \theta(n) = \frac{n+4/3}{n-16/9}.$$

Theorem

Let $\xi_i \in Z_i^{(0)}$ and let the interval sequences $(Z_i^{(m)})$ $(i=1,\ldots,n)$ be defined by the formula

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{h_i^{(m)}}{1-W_i^{(m)}} \quad (i=1,\ldots,n;\ m=0,1,\ldots). \tag{2.9}$$

Then, under the condition

$$\rho^{(0)} > 3(n-1)r^{(0)} \tag{2.10}$$

for each $i=1,\ldots,n$ and $m=0,1,\ldots$ we have

(i) $\xi_i \in Z_i^{(m)}$;

(ii) $r^{(m+1)} < \dfrac{7(n-1)r^{(m)^3}}{[\rho^{(0)}-\theta(n)r^{(0)}]^2}$.

Proof

We shall prove the first assertion by mathematical induction. Suppose that $\xi_i \in Z_i^{(m)}$ for $i \in \{1,\ldots,n\}$ and $m=0,1,\ldots$. On the basis of (2.2) and (2.9), it follows

$$\xi_i \in z_i^{(m)} - \frac{h_i^{(m)}}{1 - \sum\limits_{\substack{j=1\\ j\neq i}}^{n} \dfrac{h_j^{(m)}}{z_j^{(m)} - Z_i^{(m)}}} = Z_i^{(m+1)}.$$

Since $\xi_i \in Z_i^{(0)}$, we obtain $\xi_i \in Z_i^{(m+1)}$ for each $m=0,1,\ldots$.
Let us now prove that the interval process (2.9) has the convergence order equal to three (the second assertion). Using the properties of circular arithmetic and the inclusion (2.5), from (2.9) we obtain

$$r_i^{(m+1)} = \mathrm{rad}(Z_i^{(m+1)}) < \mathrm{rad}\left[\frac{|h_i^{(m)}|}{\{1-w_i^{(m)}; \eta^{(m)}\}}\right]$$

$$= \frac{|h_i^{(m)}|\,\eta^{(m)}}{|1-w_i^{(m)}|^2 - \eta^{(m)^2}}.$$

Using (2.7) and (2.10) we find

$$H^{(0)} < a r^{(0)},$$

$$\lambda^{(0)} < \frac{1}{3(n-1)},$$

$$\eta^{(0)} < \frac{a(n-1)r^{(0)^2}}{\rho^{(0)^2}} < \frac{a}{9(n-1)}.$$

Taking these limitations, we can write for each $i=1,\ldots,n$

$$r_i^{(1)} \leqslant r^{(1)} < \frac{H^{(0)}\eta^{(0)}}{\left[1 - \dfrac{(n-1)H^{(0)}}{\rho^{(0)}}\right]^2 - \eta^{(0)^2}}$$

$$< \frac{a^2(n-1)r^{(0)^3}}{\rho^{(0)^2}\left\{\left[1 - a(n-1)\lambda^{(0)}\right]^2 - \eta^{(0)^2}\right\}}$$

$$< \frac{a^2(n-1)r^{(0)^3}}{\rho^{(0)^2}\left[(1-\frac{a}{3})^2 - \dfrac{a^2}{81(n-1)^2}\right]} < \frac{7(n-1)r^{(0)^3}}{\rho^{(0)^2}}.$$

Hence

$$r^{(1)} < \frac{7(n-1)r^{(0)^3}}{\left[\rho^{(0)} - \theta(n)r^{(0)}\right]^2}$$

and

$$r^{(1)} < \frac{7(n-1)r^{(0)}}{\left[\dfrac{\rho^{(0)}}{r^{(0)}}\right]^2} < \frac{r^{(0)}}{\frac{9}{7}(n-1)}.$$

Starting from (2.10), in the same way as in [9], [10] or [17], it can be proved that the discs $Z_1^{(1)},\ldots,Z_n^{(1)}$ are disjoint and the following inequality

$$\rho^{(1)} > 3(n-1)r^{(1)}$$

holds.
Using the above consideration and mathematical induction, in a similar way as in [17] it is proved that the following relations are valid for each $m \in N$:

$$r^{(m+1)} < \frac{7(n-1)r^{(m)^3}}{\rho^{(m)^2}}, \tag{2.11}$$

$$\rho^{(m+1)} > 3(n-1)r^{(m+1)}, \tag{2.12}$$

$$\rho^{(m)} > \rho^{(0)} - \theta(n)r^{(0)}. \tag{2.13}$$

By virtue of (2.11) and (2.13) it follows

$$r^{(m+1)} < \frac{7(n-1)r^{(m)^3}}{\left[\rho^{(0)} - \theta(n)r^{(0)}\right]^2}$$

proving that the sequence $(r^{(m)})$ converges to zero at least cubically.
For example, for $n > 3$ we obtain the following estimate

$$r^{(m+1)} < \frac{7(n-1)r^{(m)^3}}{[\rho^{(0)} - 2.4r^{(0)}]^2}.$$

We proved earlier that (2.10) implies $\rho^{(m+1)} > 3(n-1)r^{(m+1)}$ (m=0,1,...). Therefore, lemma 2 is applicable for each m=0,1,..., so that $0 \notin 1 - W_i^{(m)}$. Therefore, under the condition (2.10), the interval process (2.9) is defined in each iterative step.
This completes the proof of the theorem. □

Remark 2

From the expression for $r^{(m)}$ it is easy to show that the convergence factor tends to $(n-1)/(\min_{i,j}|\xi_i - \xi_j|)^2$ as $m \to +\infty$. The convergence factor $7(n-1)/\rho^{(m)^2}$ is greater because the estimations and inequalities, used to prove the theorem, are not strong. Moreover, in practical application of the interval process (2.9), the quotient $\rho^{(0)}/r^{(0)}$ can be much less than $3(n-1)$ (the condition (2.10)). This is verified in many examples. If the polynomial degree n is higher, the quotient $\rho^{(0)}/r^{(0)}$ can be taken smaller. For example, for $n \geq 5$ the condition (2.10) can be replaced by the weaker condition $\rho^{(0)} > 2(n-1)r^{(0)}$. Finally, note that cubical convergence of the interval method

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{1}{\dfrac{P'(z_i^{(m)})}{P(z_i^{(m)})} - \sum_{\substack{j=1\\j\neq i}}^{n} \dfrac{1}{z_i^{(m)} - Z_j^{(m)}}} \qquad (2.14)$$

$$(i=1,...,n; m=0,1,...),$$

proposed in [7] (see, also [2, Ch. 9], [8], [10], [12, Ch. 6]), is stated under the initial condition $\rho^{(0)} > 6(n-1)r^{(0)}$. But, in [17] it is proved that the method (2.14) converges cubically under the weaker assumption $\rho^{(0)} > 2nr^{(0)}$.

Note that for the refinement of each disc by (2.14) all remaining discs are necessary. But the application of the interval method (2.9) enables the refinement of only one disc using the approximations of the remaining zeros.

In order to improve the inclusive discs for k zeros of a polynomial ($1 \leq k < n$), Gargantini considered in [7], [10] a modification of the method (2.14) with quadratic convergence. Starting data were k discs which contain these zeros and the circular region not containing the remaining n-k zeros. Now, we shall point out a simplified version of the method (2.9) with quadratic convergence also.

According to (2.1) it follows that we need not improve all approximations of zeros in each iterative step. Let $z_{k+1}^{(0)}, ..., z_n^{(0)}$ be the initial approximations which are sufficiently close to $\xi_{k+1}, ..., \xi_n$, and let $Z_i^{(0)} = \{z_i^{(0)}; r_i^{(0)}\}$ be the initial discs containing ξ_i (i=1,...,k) such that (2.10) holds. Then, we can establish the iterative interval process for the simultaneous improvement of k discs :

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{h_i^{(m)}}{1 - \left[\sum_{\substack{j=1\\j\neq i}}^{k} \dfrac{h_j^{(m)}}{z_j^{(m)} - Z_i^{(m)}} + \sum_{j=k+1}^{n} \dfrac{h_j^{(0)}}{z_j^{(0)} - Z_i^{(m)}}\right]}$$

$$(i=1,...,k; m=0,1,...). \qquad (2.15)$$

Since

$$\operatorname{rad}\left[\sum_{j=k+1}^{n} \frac{h_j^{(0)}}{z_j^{(0)} - Z_i^{(m)}}\right] = 0(r_i^{(m)}),$$

the interval process (2.15) converges quadratically. Naturally, if each $z_j^{(0)}$ (j=k+1,...,n) is closer to ξ_j and if $r_1^{(0)}, ..., r_k^{(0)}$ are less, then the sequences $(r_i^{(m)})$ (i=1,...,k) will converge faster to zero.

Remark 3

For sufficiently small $r_i^{(m)}$ from (2.9) we obtain the following approximate expression for the center $z_i^{(m+1)}$ of the interval $Z_i^{(m+1)}$:

$$z_i^{(m+1)} = z_i^{(m)} - \frac{h_i^{(m)}}{1 - \sum_{\substack{j=1\\j\neq i}}^{n} \dfrac{h_j^{(m)}}{z_j^{(m)} - z_i^{(m)}}} \qquad (2.16)$$

$$(i=1,...,n; m=0,1,...).$$

The condition "sufficiently small $r_i^{(m)}$" corresponds to the choice of the initial approximations $z_1^{(0)}, ..., z_n^{(0)}$ such that these approximations are sufficiently close to the zeros $\xi_1, ..., \xi_n$. The sequences of interval centers $(z_i^{(m)})$, defined by (2.16), converge cubically to ξ_i (i=1,...,n), which is proved in [14] (see, also [15]).

In this paper the procedure for finding the initial discs is not considered. Instead, we shall point at a simple procedure for localization of polynomial complex zeros which uses the result given by D. Braess and K. P. Hadeler [4].

Let $z_1, ..., z_n$ be sufficiently good approximations of the zeros $\xi_1, ..., \xi_n$ respectively (For finding these approximations there are many effective methods). Then, all zeros of polynomial P are contained in the union of n discs

$$G = \bigcup_{j=1}^{n} \{z \,\|z - z_j\| \leq R\},$$

where

$$R = \sum_{k=1}^{n} |h_k|.$$

If $\min_{i,j}(|z_i - z_j|) > 2R$, then all discs are disjoint and each of them contains only one zero. Namely, in such case the condition (2.4) that two discs $G_i = \{z_i; R\}$ and $G_j = \{z_j; R\}$ ($i \neq j$) being separated is satisfied.

3. NUMERICAL RESULTS

In order to test the simultaneous interval method (2.9), the program was writen in FORTRAN IV for the Honeywell 66 system. Double precision arithmetic (eighteen significant digits) was used. At the beginning of each iterative step the quantities

$h_i^{(m)} = P(z_i^{(m)})/Q'(z_i^{(m)})$ $(i=1,\ldots,n)$ were calculated, the real and the imaginary parts being calculated separately. Then, these values were used for finding the discs $Z_1^{(m)},\ldots,Z_n^{(m)}$. Practically, the initial discs, which contain the exact polynomial zeros, were chosen so that the quotient $\rho^{(0)}/r^{(0)}$ was much less than $3(n-1)$. Finding the polynomial zeros with the degree $n \leqslant 12$, this quotient was less than 4.

Example

The polynomial

$P(z) = z^7 + z^5 - 10z^4 - z^3 - z + 10$

was chosen to illustrate the interval method (2.9) numerically. The exact zeros of P are $\xi_1=2$, $\xi_{2,3}=\pm1$, $\xi_{4,5}=\pm i$, $\xi_{6,7}=-1\pm2i$. The initial discs, containing these zeros, were chosen to be $Z_i^{(0)} = \{z_i^{(0)}; 0.3\}$ where

$z_1^{(0)} = 2.2$, $z_2^{(0)} = 1.2+0.1i$, $z_3^{(0)} = -0.8-0.1i$,

$z_4^{(0)} = 0.1 + 1.2i$, $z_5^{(0)} = -0.1 - 0.8i$,

$z_6^{(0)} = -1.1 + 2.2i$, $z_7^{(0)} = -1.1 - 1.8i$.

The largest radii of discs obtained after the first and the second iteration were $r^{(1)} \cong 5.03 \times 10^{-2}$ and $r^{(2)} \cong 2.77 \times 10^{-5}$. For the third iteration we obtained the following discs :

$Z_1^{(3)} = \{2.00000000000000\underline{1}6 - 1.11 \times 10^{-17}i;\ 5.09 \times 10^{-17}\}$

$Z_2^{(3)} = \{1.00000000000000\underline{0}53 + 1.29 \times 10^{-17}i;\ 7.15 \times 10^{-16}\}$

$Z_3^{(3)} = \{-1.0000000000000000\underline{0}3 - 2.06 \times 10^{-18}i;\ 3.12 \times 10^{-17}\}$

$Z_4^{(3)} = \{3.06 \times 10^{-18} + 0.99999999999999999\underline{9}9\ i;\ 2.12 \times 10^{-17}\}$

$Z_5^{(3)} = \{1.63 \times 10^{-18} - 0.9999999999999999\underline{9}81\ i;\ 1.09 \times 10^{-16}\}$

$Z_6^{(3)} = \{-1.00000000000000000\underline{0}$
$+2.0000000000000000\underline{0}\ i;\ 3.61 \times 10^{-18}\}$

$Z_7^{(3)} = \{-1.00000000000000000\underline{0}$
$-2.0000000000000000\underline{0}\ i;\ 7.92 \times 10^{-18}\}$.

The largest radius was $r^{(3)} = r_2^{(3)} \cong 7.15 \times 10^{-16}$. The underlined digit in the above list corresponds to the order of radius.

In order to compare, many algebraic equations were solved by the interval methods (2.9) and (2.14). These examples demonstrate similar behaviour of the mentioned methods. The radii of inclusive discs, obtained by (2.9) and (2.14), have the same order of degree. Besides, we can note that :

(i) formula (2.14) requires the calculation with n–1 intervals in each iterative step, while formula (2.9) uses only one interval; in such a way it is possible to calculate an inclusive disc for arbitrary zero without using the remaining discs or the simultaneous inclusion of k discs $(1 \leqslant k \leqslant n)$ by (2.15);

(ii) the interval formula (2.14) requires a lesser number of numerical operations than (2.9).

REFERENCES

1. ABERTH O. : "Iteration methods for finding all zeros of a polynomial simultaneously", Math. Computation 27 (1973), 339-344.
2. ALEFELD G., HERZBERGER J. : *Einführung in die Intervallrechnung*, Bibliographisches Institut AG, Zürich 1974.
3. BÖRSCH-SUPAN W. : "A posteriori error bounds for the zeros of polynomials", Numer. Math. 6 (1963), 380-398.
4. BRAESS D., HADELER K. P. : "Simultaneous inclusion of the zeros of a polynomial", Numer. Math. 21 (1973), 161-165.
5. DOČEV K. : "Видоизменен метод на Ньутон за едновременно приблизително пресмуатане на всички корени на дадено алгебрично уравнение", Fiz.-Mat. Spis. B"lgar. Akad. Nauk 5 (1962), 136-139.
6. EHRLICH L. W. : "A modified Newton method for polynomials", Comm. ACM 10 (1967), 107-108.
7. GARGANTINI I., HENRICI P. : "Circular arithmetic and the determination of polynomial zeros", Numer. Math. 18 (1972), 305-320.
8. GARGANTINI I. : *Parallel algorithms for the determination of polynomial zeros*, Proc. 3rd Manitoba Conf. Numer. Math., Winnipeg 1974, 195-211.
9. GARGANTINI I. : "Parallel Laguerre iterations : complex case", Numer. Math. 26 (1976), 317-323.
10. GARGANTINI I. : "Further applications of circular arithmetic: Schroeder-like algorithms with error bounds for finding zeros of polynomials", SIAM J. Numer. Anal. 3 (1978), 497-510.
11. GLATZ G. : *Newton-Algorithmen zur Bestimmung von Polynomwurzeln unter Verwendung komplexer Kreisarithmetik*", In : Interval Mathematics (ed. K. Nickel), Berlin : Springer-Verlag 1975, 205-214.
12. HENRICI P. : *Applied and computational complex analysis*, Vol. I, Wiley-Interscience, New York 1974.
13. KERNER I. O. : "Ein Gesamtschrittverfahren zur Berechnung der Nullstellen von Polynomen", Numer. Math. 8 (1966), 290-294.
14. NOUREIN A. W. : "An iteration formula for the simultaneous determination of the zeroes of a polynomial", J. Comput. and Appl. Math. 4 (1975), 251-254.
15. NOUREIN A. W. : "An improvement on two iteration methods for simultaneous determination of the zeroes of a polynomial", Int. Comput. Math. 3 (1977), 241-252.
16. PETKOVIĆ M. S. : "Some interval methods of the second order for the simultaneous approximation of polynomial roots", Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No 634-No 677 (1979), 74-81.
17. PETKOVIĆ M. S. : "On a generalisation of the root iterations for polynomial complex zeros in circular interval arithmetic", Computing 27 (1981), 37-55.

18. PREŠIĆ S. B. : "Un procédé itératif pour la factorisation des polynomes", C. R. Acad. Sci. Paris, 262 (1966), 862-863.

19. WEIERSTRASS K. : *Neuer Beweis des Satzes, dass jede Ganze Rationale Function einer Veränderlichen dargestellt werden kann als ein Product aus Linearen Functionen derselben Veränderlichen.* Ges. Werke 3, 1903, 251-269.