Mathematical approach to ranking the greatest tennis players (GOAT)

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Abstract. The main goal of this paper is to rank the most successful tennis players to the present day, using an original mathematical method. More precisely, we are concentrating on the ranking positions of tennis greats Federer, Nadal, and Djoković (commonly known as The Big Three), who have undoubtedly dominated men's singles tennis for nearly two decades. In order to achieve as objective ranking of the mentioned trio as possible, mathematical models are used to analyze a large number of relevant tennis achievements of The Big Three, taking into account the corresponding average weights of these achievements using two approaches: scalar average weights and interval weights. We deal with the average weights equal to the sum of the ratings of all evaluators for a fixed category divided by the number of evaluators (a statistical approach). These weights aim to evaluate tennis achievements (classified in 25 categories) as adequately as possible, with respect to the importance of each of them as well as their relative relationship. Interval approach provides the self-correction if some categories are underestimated or overestimated, reducing the subjectivity in finding the weights of achievements. Both types of weights give the ranking 1. Djoković, 2. Federer, 3. Nadal, with noticeable differences in the calculated points of these trio.

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1. Introduction

The discussion about who is the best athlete of all time in his discipline has always been intriguing and very interesting for a huge number of sports fans. In these debates, the term GOAT is often used, which is an acronym for "Greatest Of All Time". This article is extended, improved, and updated (by new data, including Australian Open 2023) version of the essay published in Serbian in the magazine Nova Galaksija [1] devoted to the popularization of science. We emphasize that we are not looking for the best tennis player because that is an insufficiently defined task full of subjectivity factors, see discussion given later. Instead, the goal of this paper is to rank the most successful tennis players based on their tennis achievements, using an original mathematical method. More precise, we are concentrated on the ranking positions of tennis greats Federer, Nadal, and Djoković, who have undoubtedly dominated men's singles tennis for nearly two decades. Although the statistical approach is often used in sports (see, e.g., [2], [3]), according to the author's knowledge, the proposed method for ranking of tennis players, partly relied on statistical law of averages, appears for the first time in the literature. A different mathematical approach based on network theory was given in [4].

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Perhaps my most important motive for writing this article is the observed fact that both amateurs and professionals – the GOAT creators – very rarely consider all relevant tennis achievements, and even when they have a large amount of data, they do not take into account their levels of significance, usually known as "weight". Instead, their ranging lists rely only on individual great achievements (for example, the most number of Grand Slam titles won determines the most successful tennis player), personal impressions, style and elegance in the tennis player's game, etc. This conclusion can be reached by carefully analyzing dozens of articles by editors or commentators of renowned sports magazines and websites dedicated to tennis, as well as the statements of the tennis players themselves and their coaches. In order to achieve (or try to achieve) as objective ranking of the mentioned trio as possible, mathematical models are used to analyze a large number of relevant tennis achievements of tennis players, taking into account the corresponding weights of these achievements using two approaches: scalar weights and interval weights.

2. Pre-Open era and post-Open era: comparison

It is hard to date when the GOAT controversy became an obsession among fans and sports journalists. According to Tim Joyce, sports editor of The Guardian newspaper [5], the beginning of the career of Muhammad Ali (originally born Cassius Clay), who promoted himself frequently as the best boxer ("I am the greatest!"), is often mentioned, and later this term to be used massively appeared after the fantastic sports successes of the basketball player Michael Jordan and the golfer Tiger Woods. Following the opinion of the majority of sports experts, analysis and debates about the GOAT are not only very complicated but also unreliable because they largely depend on the subjective of the strength of the athletes' achievements and the ranking of those achievements in terms of priority (importance), where the factor of subjective assessment is (almost) impossible to eliminate. The subjectivity factor is particularly conspicuous in tennis due to the large number of various categories of achievements. Worse yet, the strength levels of those categories span a fairly wide range of numerical values. For example, 1 is an achievement of minor importance; 10 is winning a Grand Slam, the *highest reference level*.

In recent years in the world of tennis, active and former professional tennis players, coaches, sports journalists who write about tennis, and above all, thousands of fans of this game, have expressed their comments on websites, in electronic sports magazines, or on social networks, often concentrating on the question of *who is the best tennis player in the history of tennis*, the so-called GOAT. A witty but useful coincidence: in English, GOAT means goat. On websites and sports magazines, it is common to see the symbol  representing the goat in connection with the acronym. Although it has been in use for a long time, the term GOAT is awkwardly chosen because the current moment does not mean the end of history, so the terms "ever" or "in the history of tennis" (and, in general, sports) should not be used. No one can claim that a more successful tennis player will not appear in the future. Hence it follows quite clearly that, essentially, the GOAT cannot exist! The casual remark, "Well, everyone knows what is meant by the word "goat" could be just folklore, but the term GOAT is linguistically incorrect." But, unfortunately, many people do not mind (mis)using this term; for example, the GOAT stands for Greatest Of All Time, which intends to express ALL TIME literally, not only "of one specific era".

Some sportswriters sometimes forget tennis giants such as Laver, Emerson, and Rosewall, who played before the Open Era (1968). A double Grand Slam winner Rod Laver emphasizes in his monograph [6], and with whose opinion most tennis experts and players agree, there is no doubt that every era had its tennis greats. Is it possible to compare tennis players from different epochs is quite different question; a short discussion on this theme is given below. In the earlier history of the Ranking Era (since 1973), when the tennis legends Connors, Borg, McEnroe, then Lendl and Wilander, and finally Sampras and Agassi played (grouped exactly as listed, competitively), with certain periods of overlap when there were more competitors, the term GOAT rarely appeared. This selection of seven tennis players was made based on the criteria that each of them won at least seven Grand Slams (abbreviated GS) in the Open era. At the beginning of the 21st century, when Sampras retired from competition, it was said that he was the most successful tennis player of his era (with 14 Grand Slams won and 286 weeks spent at the top of the ATP list), but he was surpassed by a trio of outstanding tennis players, first Federer, and later Nadal and Djoković. According to Don Farrell [7], they have become a remarkable symbol of tennis giants. The reason why in this article we consider the magnificent trio Federer, Nadal and Djoković, often called by a common nickname the Big Three, stems from the fact that in earlier eras there was no player or group of players who was (were) significantly more successful than their competitors for a long period of time, at least in relation to the aforementioned the Big Three who dominates world tennis for almost two decades. By the way, some journalists named this trio **Fe**(derer)(Na)**dal**(Djok)**ović**, that is, **Fedalović**. Well, a fancy name.

Before further discussion, two very important questions should first be answered: 1) what does the best/ greatest athlete mean, 2) what the ranking period is it about, and 3) is it possible to compare tennis players from different epochs? These questions also have a much more global meaning. Here is a very simple but impressive example. Which car is the best: the fastest, the one with the best design, the safest, the most comfortable, or the one that consumes the least fuel, etc.? This example is given to show that individual sports disciplines are characterized by many important independent characteristics, so it is very difficult to answer which athlete is the best in his discipline, even if we limit ourselves to a specified period of time.

Regarding the first question, I would first quote Christopher Clarey [8], a New York Times sports commentator, who asked a legitimate question: if personal impressions are eliminated (the most refined and elegant play), what does it actually mean to be the best? Is it the most successful player in tennis sport, or the player who executes all the shots perfectly, including the serve and the net game, or the one who introduced new techniques, or combinations of these top achievements, etc. Clarey emphasizes that arguments for the best are subjective, often insufficiently defined, and mutually incomparable (problem of metrics). What can be measured and numerically expressed is the score of players over a specific period of time; for this reason, in this paper, our attention is focused precisely on the ranking of tennis players, analyzing only their achievements in tournaments. Given that only the categories of achievements of players can be measured by a unique metric and expressed numerically, it follows that debates about the more successful tennis player, limited to one specific epoch, are only achievable and thus more acceptable than insufficiently defined search for the best player. This is the reason that in this paper, the attention is focused on the ranking of tennis players analyzing only their achievements in tournaments.

The abbreviation GOTOG (Greatest of Their Own Generation) of one specific era may have been forcibly constructed, but at least we have no doubts about which period of time is in question; besides, we will not dream of goats. To avoid linguistic gymnastics and weird abbreviations, we will tacitly retain the term "greatest" (the greatest, the most famous, etc.), but in the sense of the most successful. It is more precise to say MSOTOG (The Most Successful of Their Own Generation), but this linguistic abbreviation would unnecessarily burden the text.

The second and third questions about comparing different epochs and the strength of tennis players from these epochs are hypothetical, and a serious discussion could only be conducted assuming that tennis players from the pre-Federer-Nadal-Djoković era had the conditions and equipment that exist today, while preserving their extraordinary talent and outstanding tennis skills. Would they be equal partners to today's greats, or would one of the generations be ahead? Sports journalist Bruce Miller humorously stated in the daily San Diego Union Tribune: "The sport, then and now, is as stylistically different as Bob Hope and Bob Dylan" [9]. Many authors of the texts on the subject of the just-mentioned comparisons cautiously declare that they are not fans of these "what-if " fantasies since predictions about which generation would win are in the realm of conjecture, speculation, and too many assumptions. What is pretty certain is that tennis has improved a lot. Most of the top tennis players of the pre-Federer-Nadal-Djoković era and eminent editors and commentators of renowned sports magazines and websites (among them World Tennis Magazine, Tennis Magazines Newsletter, Tennis Threads, Tennis Industry Magazine, Inside Tennis Magazines, tennis.com, ultimatetennisstatistics.com, tennisespresso.com) express the opinion that there were great tennis players in earlier generations, but in terms of success in tournaments, the players of today's era would have a great advantage. In his autobiography [6] Laver wrote: "The players of the current era hit the ball at 135 mph. We could never generate much more than 105 mph. The topspin those players can put on their forehand, it is unbelievable." This is made possible by perfected graphite rackets made of graphite combined with titanium, hypercarbon, and other materials and with polyester strings, enabling topspin and other types of perfect shots. In this regard, we quote the words of Eurosport commentator Mats Wilander, an outstanding player from the period 1982–1988 and the winner of 7 Grand Slams: "Sometimes as a commentator, I criticize some actions of Federer, Nadal, and Djoković, even though in a match against them I would hardly win any game playing like in my best days."

Finally, we present comparisons of sports experts' opinions taken from various sports journals and websites about the styles of tennis play Then-and-Now. Once the style of the game was based on the serve-and-volley game, today tennis players from the baseline with strong and precise return shots with topspins would "smash" a tennis player with a serve-and-volley game in a large percentage. Today, daily training sessions of several hours for the acquisition of physical and mental condition enable a strong pace of play and readiness for very long matches, even up to five hours, and for faster movement and better footwork, which results in returning very difficult balls; perfect athletic ability is one of the main characteristics of the game of modern tennis players. In the pre-Open era, tennis players' physical conditioning used to be timed for short serve-and-volley matches; in addition, mental-psychological preparations and training were modest. The mentioned advantages enable today's top tennis players to have different playing styles and a sense of changing tactics, thus achieving a "balance of excellence" during the game, especially in critical situations, which was not the case in previous eras.

Although sports journalists and tennis experts offer cautious comments on the topic just described, they cannot resist providing their rankings with lengthy arguments and comments. We list some of the "fresh" lists, where, again, the Fedalović trio dominates.

How They Play 1. Djoković 2. Nadal 3. Federer 4. Laver 5. Sampras
`https://howtheyplay.com/individual-sports/Top 10-Greatest-Male-Tennis-Players-of-All-Time`

Ultimate tennis statistics 1. Djoković 2. Federer 3. Nadal
`https://www.ultimatetennisstatistics.com/goatList`

Sporting News 1. Federer 2. Djoković 3. Nadal 4. Connors 5. Sampras
`https://www.sportingnews.com/au/tennis/news/wimbledon-2022-mens-tennis-goat-rank-federer-nadal-djokovic/bhqliho5migps1wghxs7qq1i`

Tennis Espresso 1. Djoković 2. Federer 3. Nadal
`https://www.tennisespressojournal.com/`

We notice that the Big Three occupy the first three places in all presented lists.

In addition, we point out one respectable fact about the winners of the Grand Slams (certainly the greatest achievement in tennis) from the group of seven mentioned tennis players from previous eras:

Pre-Fedalović era: Connors (8 Grand Slams won), Borg (11), McEnroe (7), Wilander (7), Lendl (8), Agassi (7), Sampras (14) (1974–2003) Total: **62**

Fedalović era: Nadal (22), Djoković (22), Federer (20) Total: **64**

The very fact that Nadal, Djoković and Federer together have won more Grand Slam tournaments than seven tennis players from the 1974-2003 era, as well as other data that can be seen on tennis websites (see also the List of Achievement Rankings given below), clearly says that the current generation of tennis players is undoubtedly more successful (according to the discussions above, we deliberately avoid the word better) than any generation of tennis players before them. Having in mind the above Then-and-Now discussion as well as great respect for the achievements of all previous generations of tennis players, today the vast majority of sports analysts are of the opinion that the ranking of participants in individual sports (tennis, chess, golf, boxing, athletics, etc.) should be divided into two categories:

– The most successful tennis player to these days;

– The most successful tennis player nowadays.

According to the above discussion about the comparison of tennis players belonging various generations, almost exclusively based on their achievements, the answer to the question of who is the greatest player of the current generation (Fedalović' era) is also answer with great probability to the question who is the greatest player until these days; certainly not the ultimate answer but an answer with a fair degree of reliability. Indeed, it is very difficult to find arguments that somebody can be the most successful tennis player to the present day without being the most successful tennis player of the era that is currently ongoing.

Remark 1. For a large number of tennis experts, tennis players, and tennis fans , it is acceptable to say that the aforementioned Fedalović is the most successful tennis player of today's generation. This coincides with the opinion of the tennis legend John McEnroe, who believes that all three members of the Big Three, in one way or another, are GOTOG [10]. This McEnroe conclusion is perhaps the closest to the proper answer we are looking for (here we use the term GOTOG instead of GOAT for the aforementioned reason that "the best ever" cannot exist at the present moment).

3. The question of importance of the achievements

As mentioned at the beginning, one of the most difficult problems in searching for the most successful tennis player, either of the current epoch or the Open era to the present day, is the determination of proper estimate of the importance degree of the athletes' achievements, shorter the *weight*. The main reason is the presence of the factor of subjectivity in the process of evaluating the level of importance of the categories, which is inevitably emphasized by almost all sports experts and tennis players.

In order to obtain the most objective ranking of the members of the Big Three, an original mathematical model is proposed in this contribution that uses a large number of relevant tennis achievements of Nadal, Federer, and Djoković during their careers as input data. In addition to numerical data describing achievements, the corresponding weights of importance of these achievements (categories) are introduced. These weights aim to evaluate tennis achievements (classified in 25 categories) as adequately as possible, taking into account the importance of each of them as well as their relative relationship. For example, a Grand Slam title is certainly about twice as valuable as a Masters 1000 title, the number of weeks at the top of the ATP list is a more significant achievement than the number of weeks spent in the Top 10, etc. A very favorable circumstance for this ranking is that Nadal, Djoković and (the recently retired) Federer played in (almost) the same time period, so the comparison of their achievements makes a lot of sense. Let us emphasize that the presented mathematical model can be applied to a group of an arbitrary number of players.

Speaking about the weight of a category, we assume the average weight calculated using some basic elements of statistics. According to the Law of Large Numbers (LLN) (commonly called the *law of averages*), one of the fundamental laws in the Theory of Probability and Statistics that describes the result of performing the same experiment a large number of times, the objectivity of category rating (that is, the determination of weight) would be more reliable if as many analysts as possible were involved in the rating action, and then dealing with the average rating (= the sum of the ratings of all particular weights for a fixed category divided by the number of evaluators). Mathematically, if M is the number of evaluators and $w_{1,i}, \dots, w_{M,i}$ are their proposed weights for the fixed category i, then the average weight $W_{a,i}$ is

$$W_{a,i} = \frac{w_{1,i} + \cdots + w_{M,i}}{M}.$$

The subscript index a of the capital W points out to the average weight.

Although the number of evaluators in this project was relatively large ($M = 30$), the conditions for the application of the Law of Large Numbers were not sufficiently met, but certainly this approach (in order to reduce the subjective factor) is significantly better compared to the evaluations of, say, 4 to 5 evaluators. It is clear that the effect of reducing the degree of subjectivity would be greater if the number of evaluators would be greater; consequently, the ranking list of the most successful tennis players would be more realistic. At the same time, when determining new average weights, following the principle used in statistical analysis, all weights are discarded if they deviate extremely from the mean value $W_{a,i}$ (in each category i). If $S_{k_i}^{(i)}$ denotes the sum of all discarded weights (total k_i) in category i, then the new average weight is calculated by the formula,

$$\widetilde{W}_{a,i} = \frac{M \cdot W_{a,i} - \big(S_{k_1}^{(1)} + \cdots + S_{k_{25}}^{(25)}\big)}{M - \big(k_1 + \cdots + k_{25}\big)}.$$

Since the described determination of the average weight $\widetilde{W}_{a,i}$ requires vast number of evaluators (mission impossible within a small project), the author was forced to deal with the simplified average weight $W_{a,i}$.

4. List of tennis achievements of the Big Three

When ranking in this contribution, the author considered 25 tennis achievements (categories) of Nadal, Federer and Djoković during their careers. This selection of categories is based on characteristics that can be described with numerical data and on their most frequent appearance on numerous websites and tennis magazines (some of them mentioned above). Certainly there are some other categories that characterize the qualities and achievements of these tennis players, but all of them are either difficult or impossible to represent numerically. Virtuosity in the game, outstanding forehand and backhand shots, great returns, elusive spin shots, 'killer' serves, volley shots, excellent net-play, mental strength, athleticism, etc. are characteristics that all together could be classified as a superior style of play that brings a tennis player points, and this is implicitly expressed through the numerical data given in the List of the achievements of the three mentioned tennis players.

Below is a list of categories along with the achievement weights of Nadal, Federer, and Djoković (expressed numerically), using the abbreviations N, F, and D, respectively. The average weighs $W_{a,i}$ $(i = 1, 2, \dots, 25)$ are also given, ranging from 1 to 10, whereby the title won at the Grand Slam is scored with the highest score of 10, as a reference category. Ratings are rounded to X.0 (the whole number) or X.5 (the whole number + 0.5). Note that the list of categories was formed using several relevant sources, but small errors are still possible. Nevertheless, they have a completely negligible influence in the entire process of applying the mathematical model.

The categories are classified by areas that are approximately of the same type, with the exception of area E, which contains several specific categories. The input data in B and C give the ratio of titles won or individual wins vs. the total number of Grand Slams, Masters, and matches played. Here, proper fractions were kept instead of decimal numbers in order to have an insight into the titles won (individual wins). For example, in category 9, the input data is given in the form of 20/37 instead of 0.54..., although the calculation continues with a decimal number. Note that *title* means the tournaments won, while *win* means the victory in one match, player vs. player.

List of tennis achievements, last update: January 30, 2023

A: Number of titles won at the ATP tournaments

1. Number of titles-Grand Slam tournaments:
   – N=22, F=20, D=22; • $W_{a,1} = 10$ - reference rating
2. Number of titles-final Masters tournaments 1500:
   – N=0, F=6, D=6; • $W_{a,2} = 6$
3. Number of titles-Masters 1000 tournaments:
   – N=36, F=28, D=38; • $W_{a,3} = 5.5$
4. Total number of titles at the ATP tournaments:
   – N=92, F=103, D=93; • $W_{a,4} = 4.5$

B: Number of titles/total number of tournaments

5. Grand Slam title won/total number of Grand Slams played:
   – N=22/66, F=20/81, D=22/69; • $W_{a,5} = 6$
6. Masters 1500 titles won/total number of Masters 1500 played:
   – N=0/11, F=6/17, D=6/15; • $W_{a,6} = 4$
7. Masters 1000 titles won/total number of Masters 1000 played:
   – N=36/128, F=28/138, D=38/123; • $W_{a,7} = 3.5$

C: Number of individual wins/total number of matches

8. Number of individual GS wins/total number of GS matches:
   – N=311/353, F=369/429, D=341/388; • $W_{a,8} = 5$
9. Number of individual Masters 1500 wins/total number of matches on Masters 1500:
   – N=20/34, F=59/77, D=46/63; • $W_{a,9} = 3.5$
10. Number of individual Masters 1000 wins/total number of matches Masters 1000:
   – N=406/494, F=381/489, D=385/469; • $W_{a,10} = 3$
11. Number of individual wins in ATP tournaments/total number of matches in ATP tournaments:
   – N=1065/1280, F=1251/1526, D= 1072/1232; • $W_{a,11} = 3$
12. Number of wins vs. ATP Top 10 players/total number of matches played vs. ATP Top-10 players:
   – N=575/743, F=654/873, D=564/729; • $W_{a,12} = 3$
13. Number of wins in Davis cup/total number of matches played in Davis cup:
   – N=37/42, F=52/70, D=42/55; • $W_{a,13} = 2$
14. Number of wins in the ATP Cup/total number of matches played in the ATP Cup:
   – N=4/6, F=0/0, D=8/8; • $W_{a,14} = 1.5$

D: Numbers of weeks as World ATP No. 1

15. Number of weeks as World ATP No. 1:
   – N=209, F=310, D=374; • $W_{a,15} = 6.5$
16. The most consecutive weeks as World ATP No. 1:
   – N=56, F=237, D=122; • $W_{a,16} = 5$
17. Number of weeks among Top 10 of ATP list:
   – N=903, F=968, D=771; • $W_{a,17} = 5$

E: Miscellaneous achievements

18. The Big Three head-to-head (outcomes of N, F and D playing each other):
   – N:F=24:16, N:D=29:30, D:F=27:23; • $W_{18} = 6$
19. Number of Grand Slam finals:
   – N=30, F=31, D=33; • $W_{a,19} = 5$
20. Maximum number of points achieved as Number 1 on the ATP list:
   – N=15390, F=15903, D=16950; • $W_{a,20} = 4$
21. Number of years a tennis player has finished at the top of the ATP list:
   – N=5, F=5, D=7; • $W_{a,21} = 5$
22. Medals at the Olympic Games (gold = 8, silver = 4, bronze = 1 point)
   – N=1 gold, F=1 silver, D=1 bronze; • $W_{a,22} = 4$

23. Non-Calendar Year Grand Slam (Remark 3):
    - N=0, F=0, D=1; • $W_{a,23} = 5$

24. Most consecutive wins:
    - N=32, F=42, D=43; • $W_{a,24} = 4$

25. Double-completed Masters 1000 (each of the 9 Masters won at least twice) (Remark 2):
    - N=0, F=0, D=2; • $W_{a,25} = 6$

Remark 2. Nadal has not won any of the Miami Open or Paris Open, Federer has not won any of the Monte Carlo Open or Rome Open.

Remark 3. In this paper we use the term Grand Slam both for winning all four tournaments in the same year, Australian Open, Roland Garros, Wimbledon, US Open (the calendar year Grand Slam), as well as for winning any of these four tournaments. If necessary, the first achievement is underlined. Note that the calendar year Grand Slam has not been won by any tennis player in the ATP Rating era (from 1973 to the present day). The non-calendar year Grand Slam, category 23, pointed to winning all four Grand Slam tournaments consecutively, but not in the same calendar year. The only non-calendar Grand Slam in the ATP Rating era was won by Djoković; from Wimbledon 2015 to French Open 2016.

5. Calculation of ranking vectors

Analyzing the achieved results given in the descriptions of all categories, with the exception of category 18, we observe that a relative proportional relationship can be established between the corresponding numerical data and form the normalized *ranking vectors* $\boldsymbol{r}_i = (r_{N,i}, r_{F,i}, r_{D,i})$ $(i = 1, ..., 25)$. For example, for the number of weeks as World ATP No. 1 (category 15) we have N=209, F=310, D=374 so that

$$\mathrm{N/D} = 0.55882, \quad \mathrm{F/D} = 0.82887.$$

The ordered triple of numbers

$$\boldsymbol{r}_{15} = (r_{N,15}, r_{F,15}, r_{D,15}) = (0.55882, 0.82887, 1)$$

creates the ranking vector $\boldsymbol{r}_1$, which will be used in the final ranking of the three tennis players. In the same way, except for the category 18, we find the ranking vectors $\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_{14}, \boldsymbol{r}_{16}, \boldsymbol{r}_{17}, \boldsymbol{r}_{19}, ..., \boldsymbol{r}_{25}$.

5a Ranking by Keener's method

Category 18 (head-to-head scores between Nadal, Federer, and Djoković), which is of great importance, is specific and requires a different approach because the above method (the proportional model) of calculating the ranking vectors $\boldsymbol{r}_i$ $(i \neq 18)$ cannot be applied. In this case, the method of American mathematician James Keener [11] was applied. Keener's method can be applied not only for ranking teams in American football (the basic aim of Keener's paper), but also in many other disciplines because it has proven to be one of the most objective in practice. The determination of the optimal schedule of tasks in the design and development of new products, solving logistical problems, and hardware and software development, are only a few examples of the application of Keener's method.

Keener's approach is mathematical in nature, so unreliable results and conclusions arising from subjective impressions and assessments, are avoided to a great extent. This method requires the knowledge of some elements of Matrix algebra, so the readers who are not familiar with this subject (which is studied at technical and mathematical faculties) can skip the mathematical details and catch the connection somewhere around Example 1. For the readers who know the basic elements of matrix algebra, for completeness, we briefly describe the ranking algorithm based on Keener's method.

Let $U_1, \ldots, U_n$ be participants in a game (engineering design, competition, etc.), the outcome of which depends on U_i's achievement. It is assumed that there are interactions between the participants (who play each other). Let a_{ij} be non-negative numbers that depend on the outcome of the game between participant i and participant j. The square scheme of numbers (matrix)

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & & a_{nn} \end{bmatrix} = (a_{ij})$$

is called the *preference matrix*. It is often written in abbreviated form as $A = (a_{ij})$. The order of this matrix is equal to $n \times n$ or just n in the case of a square matrix. The vector $\boldsymbol{x}$ having n elements $x_1, x_2, \ldots, x_n$, respectively, is written over an ordered n-tuple, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. The task consists in finding a ranking model that will give as realistic a ranking of players as possible, taking into account the tennis achievements of tennis players and the mutual relationship between these achievements.

Definition. A non-zero vector $\boldsymbol{x}$ is called an *eigenvector* of a given matrix A of order $n \times n$ if there exists a real or complex number t such that

$$A\boldsymbol{x} = t\boldsymbol{x}. \tag{1}$$

The number t is called the *eigenvalue* of the matrix A, which corresponds to the eigenvector $\boldsymbol{x}$. Equation (1) has n eigenvalues $t_1, \ldots, t_n$ and each eigenvector x_j corresponds to an eigenvalue t_j. The maximum of the absolute values of the eigenvalues of the matrix A is called the spectral radius, in the notation

$$s(A) = \max\{|t_1|, |t_2|, \ldots, |t_n|\}.$$

Nonnegative matrices (having nonnegative elements) and non-negative irreducible square matrices $A = (a_{ij})$ of order n are of particular interest for the application of Keener's method. The notion of irreducibility is much more complex, and we will not deal with it because in this project we only work with square non-negative irreducible matrices, which means that the conditions for the theorem stated below are satisfied.

Keener's method is essentially based on the Perron-Frobenius theorem, one of the most important theorems in matrix algebra.

Theorem (Perron-Frobenius). *Let A be a non-negative and irreducible matrix of order n. Then*

(i) A has a positive real eigenvalue equal to its spectral radius.

(ii) $s(A)$ is a simple eigenvalue of the matrix A.

(iii) A positive eigenvector corresponds to the spectral radius $s(A)$.

Applying Keener's method, the value level of strength (*strength* in the sequel) r_j of participant j is evaluated and then forms a ranking vector $\boldsymbol{r} = (r_1, \ldots, r_n)$. The elements of the ranking vector $\boldsymbol{r}$ determine the order of participants in the ranking list.

The strength of the participant's achievements is defined by the sum

$$v_i = \sum_{j=1}^{n} a_{ij} r_j \,.$$

The position of each participant in the ranking list ant should be proportional to his achievement, that is,

$$A\boldsymbol{r} = t\boldsymbol{r} \,. \tag{2}$$

Since the relation (2) is equivalent to relation (1) and the Perron-Frobenius theorem holds, it follows that the positive eigenvector of the non-negative matrix A (corresponding to the unique spectral radius $s(A)$) just gives the ranking vector, thus the ranking question is completely solved.

Ranking algorithm:

(1) Form the non-negative matrix $A = (a_{ij})$ based on the input data;

(2) Determine the eigenvectors of the matrix A and extract the unique nonnegative eigenvector $(r_{i_1}, r_{i_2}, \ldots r_{i_n})$ whose components give the ranking list starting with the dominant component r_{ik}, i.e.

$$r_{i_k} > r_{i_m} \geq \cdots \geq r_{i_n} \,.$$

Remark 4. It is desirable to normalize the extracted ranking vector by taking the largest component r_{i_k} to be equal to 1. Then $r_{i_j} < 1 = r_{i_k}$ for every j different from k. In this way, it is possible to perceive the degree of dominance of the r_{i_k} component.

In order to apply the ranking algorithm, it is first necessary to form the preference matrix $A = (a_{ij})$ and then apply one of the algorithms for calculating the eigenvectors. If the reader is not familiar with computer algebra software such as *Mathematica, Matlab,* or *Maple*, it is best to consult experts who can do it with a few commands.

Remark 5. The software *Mathematica* (developed by Wolfram Inc.) finds the eigenvectors B of the matrix A with a simple command,

$$B = \texttt{Eigenvectors}[A]$$

which returns a matrix B composed of n vectors. This matrix always contains a unique vector whose elements are all of the same sign, and this is, actually, the ranking vector. When applying the software Mathematica, it may happen that the elements of the ranking vector are all non-positive. Based on (1) (or (2)), then we can multiply this vector by -1 and get a unique non-negative eigenvector $(r_{i_1}, r_{i_2}, \ldots, r_{i_n})$ which represents the ranking vector.

In this article, the application of the software package *Mathematica* is demonstrated using the following example:

Example 1. For the tennis players N, F, and D (we are talking about Nadal, Federer, and Djoković and their mutual matches, category 18) the input data scheme and the corresponding matrix are given by

	N	F	D
N	□	24	29
F	16	□	23
D	30	27	□

$$A_{18} = \begin{bmatrix} 0 & 24 & 29 \\ 16 & 0 & 23 \\ 30 & 27 & 0 \end{bmatrix}.$$

This matrix is loaded into *Mathematica* program as $A = \{\{0, 24, 29\}, \{16, 0, 23\}, \{30, 27, 0\}\}$. Command

$$B = \texttt{Eigenvector}[A]$$

returns the matrix B of the eigenvectors of the matrix A,

$$B = \begin{bmatrix} 0.958323 & 0.772665 & 1. \\ -0.613328 & -0.436758 & 1. \\ 1.61968 & -2.51888 & 1. \end{bmatrix}.$$

We note that the only eigenvector with elements of the same sign is represented by the first row of the matrix B, i.e.

$$C = (\mathrm{N}, \mathrm{F}, \mathrm{D}) = (0.958323,\ 0.772665\,1).$$

This vector serves to rank the participants. According to the values of the elements of the vector C, the order of participants is as follows

1. **Djoković** (1.)
2. **Nadal** (0.958323)
3. **Federer** (0.772665).

In this case the ranking vector is already normalized. In general, if one or more elements of the ranking vector C are greater than 1, then the normalization would be performed by

$$C = \frac{C}{\texttt{Max}[C]}\ .$$

Let us note that in this head-to-head category, Djoković has an advantage over Federer of approximately 23%.

6. Ranking by scalar weights

The normalized ranking vectors for all 25 categories are given in Table 1 where the simplified average weights $W_{a,i}$ were used. According to the introduced notation, $r_{N,i}$ is the number in the second column of Table 1 (referring to Nadal), $r_{F,i}$ is the number in the third column (referring to Federer), and $r_{D,i}$ is the number in the fourth column (referring to Djoković), where i marks the categories ($i = 1, ..., 25$), see Table 1.

i	$r_{N,i}$	$r_{F,i}$	$r_{D,i}$
1	1.	0.909091	1.
2	0.	1.	1.
3	0.947368	0.736842	1.
4	0.893204	1.	0.902913
5	1.	0.740741	0.913043
6	0.	0.882353	1.
7	0.910362	0.656751	1.
8	1.	0.9763	0.997555
9	0.801627	0.994474	1.
10	1.	0.948019	0.998824
11	0.963997	0.949794	1.
12	1.	0.9589	0.990288
13	1.	0.862069	0.873563
14	0.666667	0.	1.
15	0.558824	0.828877	1.
16	0.236287	1.	0.514768
17	0.932851	1.	0.796488
18	0.958322	0.772664	1.
19	0.909091	0.939394	1.
20	0.907965	0.93823	1.
21	0.714286	0.714286	1.
22	1.	0.5	0.125
23	0.	0.	1.
24	0.744186	0.976744	1.
25	0.	0.	1.

Table 1 Ranking vectors of all 25 categories

The total numerical strength N, F, and D for each of the three tennis players is obtained as the sum of the product of ranking numbers (Table 1) and simplified average weights $W_{a,i}$:

$$\begin{aligned} \mathrm{N} &= r_{N,1} \cdot W_{a,1} + \cdots + r_{N,25} \cdot W_{a,25} \\ \mathrm{F} &= r_{F,1} \cdot W_{a,1} + \cdots + r_{F,25} \cdot W_{a,25} \\ \mathrm{D} &= r_{D,1} \cdot W_{a,1} + \cdots + r_{D,25} \cdot W_{a,25}. \end{aligned} \tag{3}$$

Using the scalar weights of the categories given by the weights $W_{a,i}$ and the formula (3), the following total numerical strengths of the three tennis players are obtained:

$$\mathrm{N} = 82.07,\ \mathrm{F} = 90.52,\ \mathrm{D} = 107.80$$

or, in the normalized form,

$$\mathrm{N} = 0.761,\ \mathrm{F} = 0.840,\ \mathrm{D} = 1.$$

Therefore, based on the scalar weights, one obtains the following order of the Big Three:

1. **Djoković** (total numerical strength is equal to 1)
2. **Federer** (0.840)
3. **Nadal** (0.761)

Table 2. Ranking of the Big Three based on the scalar weights of the categories

Realistically, the results are not surprising taking into account Djokovic's tennis achievements. From Table 2 we can see that Djoković's advantage over Federer is around 16% and his advantage over Nadal is as much as 24%. Both tennis players, Djoković and Nadal, won the same number of Grand Slams (22). However, it must not be ignored the facts that Nadal has a negative score of 29:30 with Djoković, significantly fewer weeks (209) at the top of the ATP list compared to Djoković's absolute record of 374 weeks. Furthermore, Nadal has two Masters 1000 fewer than Djoković (36:38), and he does not have any final Masters 1500, while Djoković has six. None of the tennis players has achieved a calendar Grand Slam, but in the case of a non-calendar Grand Slam, only Djoković managed to do so in the past 50 years. Djoković has the best result in 16 out of 25 categories (sharing category 2 with Nadal - the number of Grand Slams won), which is not necessarily a strong argument because it depends on the choice of categories, but it still seems impressive. Let us also note that Djoković has "nonzero" achievements in all categories, unlike Federer and Nadal. However, being 24% behind Djoković and 12% behind Federer leaves no doubt that Nadal occupies the third position among the three greats of the current generation. Even if some categories, where Nadal and Federer are dominant, get better (but reasonably small) marks, it does not change the positions given by Table 2. For example, according to the calculated points (Table 1), one Grand Slam more would bring about only 1.1 percent more in the total score of any of these three players. This means that at this moment Djoković has a big advantage over Federer and Nadal as the most successful player.

7. Ranking by interval weights

In order to further reduce the subjectivity factor when determining the weighting factors, after calculating average weights $W_{a,i}$ in the first step, instead of scalar average weights we have introduced an original method dealing with the interval weights $[W_{a,i}-p, W_{a,i}+p]$, where p is the semi-width of the intervals. For example, if the mean score is $W_{a,i}=5$ and $p=0.5$, the interval weight is $[5-0.5, 5+0.5]=[4.5, 5.5]$. In this way, if the evaluator has a dilemma about estimating a category with either 4.7 or 5.2, he will cover both estimates by the interval [4.5,5.5] and thus remove the dilemma. Furthermore, interval approach provides the *self-correction* if some categories are underestimated or overestimated. In the presented analysis, the semi-width $p=0.5$ was used, which gives a sufficiently wide range of intervals equal to 1. Details of interval computations can be found in the books of Ramon E. Moore or in the monograph [12]. As noted above, interval weights $[W_{a,i}-0.5, W_{a,i}+0.5]$ $(i=1,\ldots,25)$ were used in the formula (3) instead of scalar weight factors. They are represented by the interval vector:

$$\begin{aligned}\boldsymbol{W} \;=\; & ([10,10],[5.5,6,5],[5,6],[4,5],[5.5,6.5],[3.5,4,5],[3,4],[4.5,5.5],[3,4],\\ & [2.5,3.5],[2.5,3.5],[2.5,3.5],[1.5,2.5],[1,2],[6,7],[4.5,5.5],[4.5,5.5],[5.5,6.5],\\ & [5.5,6.5],[4.5,5.5],[3.5,4.5],[4.5,5.5],[4.5,5.5],[3.5,4.5],[5.5,6.5]).\end{aligned}$$

The formula (3) gives the intervals $\boldsymbol{N}$, $\boldsymbol{F}$, and $\boldsymbol{D}$by the modified formula

$$\begin{aligned}\boldsymbol{N} &= r_{N,1}\cdot[W_{a,1}-0.5, W_{a,1}+0.5]+\cdots+r_{N,25}\cdot[W_{a,25}-0.5, W_{a,25}+0.5]\\ \boldsymbol{F} &= r_{F,1}\cdot[W_{a,1}-0.5, W_{a,1}+0.5]+\cdots+r_{F,25}\cdot[W_{a,25}-0.5, W_{a,25}+0.5]\\ \boldsymbol{D} &= r_{D,1}\cdot[W_{a,1}-0.5, W_{a,1}+0.5]+\cdots+r_{D,25}\cdot[W_{a,25}-0.5, W_{a,25}+0.5],\end{aligned}$$

or, since we deal with positive numbers,

$$
\begin{aligned}
\boldsymbol{N} &= [r_{N,1}(W_{a,1}-0.5), r_{N,1}\ (W_{a,1}+0.5)] + \cdots + [r_{N,25}(W_{a,25}-0.5), r_{N,25}(W_{s,25}+0.5)] \\
\boldsymbol{F} &= [r_{F,1}(W_{a,1}-0.5), r_{F,1}(W_{a,1}+0.5)] + \cdots + [r_{F,25}(W_{a,25}-0.5), r_{F,25}(W_{s,25}+0.5)] \quad (4) \\
\boldsymbol{D} &= [r_{D,1}(W_{a,1}-0.5), r_{D,1}(W_{a,1}+0.5)] + \cdots + [r_{D,25}(W_{a,25}-0.5), r_{D,25}(W_{a,25}+0.5)] \,.
\end{aligned}
$$

Summing the intervals, from the formula (4) the total interval strengths $\boldsymbol{N}$, $\boldsymbol{F}$, and $\boldsymbol{D}$, for each of the three tennis players are obtained in the form of intervals

$$
\begin{aligned}
\boldsymbol{N} &= [73.50, 90.64] \\
\boldsymbol{F} &= [81.34, 99.71] \\
\boldsymbol{D} &= [96.74, 118.86].
\end{aligned}
$$

The results of this interval analysis are presented graphically in Figure 1.

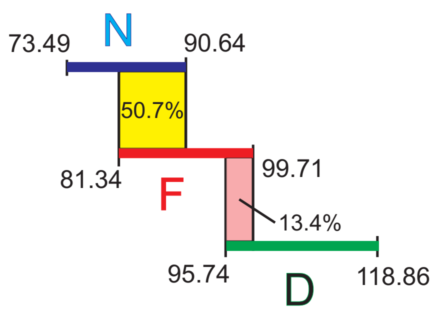

Figure 1. Achievement ranges of Nadal, Federer and Djoković

Interval mathematics, introduced to reduce the influence of subjectivity when determining the weights of categories, even more convincingly shows the dominance of Djoković from a mathematical point of view, especially in relation to Nadal, whose interval of achievements does not even intersect with Novak Djoković's interval of achievements. Federer and Djoković share 13.4%, while Federer and Nadal share 50.7% of tennis achievements.

Remark 6. Considerable differences expressed in percentages (Table 1) and lengths of overlapping intervals (Figure 1) suggest that the order of ranked players is to a good extent, acceptable from the mathematical point of view. This points to the fact that if the weights vary within a relatively small range of values, the displayed order of the members of the Big Three would hardly change. Based on the above analyzes and conclusions given in Table 2 and Figure 1, it can be concluded that the outcome of the applied mathematical method is as follows: Djoković is the most successful tennis player of his generation (GOTOG). Taking into account the above discussions, the results of the applied mathematical model (which replace the personal impressions of tennis experts and fans), and the Then-and-Now comparison, it is very close to the truth that Djoković is not only the most successful tennis player of today's generation but also the most successful tennis player to the present day. Nevertheless, the author is aware that mathematics generally cannot always provide an ideal solution to problems that are not purely mathematical in nature. For this reason, the author leaves room for discussion and debate that could lead to possible improvements of the presented ranking model. Given that the accomplished achievements, expressed numerically, cannot change for the observed epoch, the possibility of improvement mainly refers to increasing the degree of objectivity when assessing the importance of categories. As already mentioned, this can be achieved by significantly increasing the number of analysts-evaluators, together with use of the above-mentioned statistical approach in finding the average weights. Involving a large number of evaluators is a difficult undertaking that requires not only great efforts in correspondence but also the goodwill of tennis experts. It is obvious that one person cannot realize such an action alone, so the presented article, apart from the original basic approach, also has a motivational character with the aim of attracting the attention of as many tennis experts as possible who can give a great contribution in a project of a much wider scale.

8. Conclusion

The main goal of this paper is to rank the most successful tennis players to the present day, using an original mathematical method. The proposed method is applied to the ranking positions of tennis greats Federer, Nadal, and Djoković, considering 25 relevant tennis achievements of these trio together with the corresponding average weights of these achievements of the two type: scalar average weights and interval weights. Both types of weights give the ranking 1. Djoković, 2. Federer, 3. Nadal, with noticeable differences in the calculated points of these trio. Based on the scalar weights, one obtains the following order of the Big Three:

1. **Djoković** (total numerical strength is equal to 1)
2. **Federer** (0.848) (−16%)
3. **Nadal** (0.766) (−24%)

The use of interval weights gives the ranking list expressed by intervals of achievement points (formula (4)):

1. **Djoković** [96.74 , 118.86]
2. **Federer** [81.34 , 99.71]
3. **Nadal** [73.49 , 90.64].

Interval mathematics, introduced to reduce the influence of subjectivity when determining the weights of categories, even more convincingly shows the dominance of Novak Djoković.

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